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A nonconforming framework for finite element exterior calculus

SHUO ZHANG*
State Key Laboratory of Mathematical Sciences (SKLMS) and State Key Laboratory of Scientific and Engineering Computing (LSEC), Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing, 100190, China

School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, 100049, China
*Corresponding author: szhang@lsec.cc.ac.cn

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This paper presents a theory of nonconforming finite element exterior calculus based on a unified family of nonconforming finite element spaces for $H\Lambda^k$ in $\mathbb{R}^n$ ($0 \leqslant k \leqslant n$, $n \geqslant 1$), which are constructed in this paper by a novel approach that seeks to mimic the dual connections between adjoint operators. The family each employs piecewise Whitney forms as shape functions, including the lowest-degree Crouzeix-Raviart element space for $H\Lambda^0$, and optimal approximations and uniform discrete Poincaré inequalities are presented. Further, with these newly constructed finite element spaces, discrete de Rham complexes with commutative diagrams, and the discrete Helmholtz decomposition and Hodge decomposition for piecewise constant spaces are established, based on which the Poincaré-Leftschetz duality can be reconstructed discretely as an equality. The consequent framework of nonconforming finite element exterior calculus is naturally connected to the classical conforming one but significantly different. Notably, all discrete operators involved are local, namely acting cell by cell separately. The newly constructed finite element spaces do not fit Ciarlet's finite element definition, though, they admit locally supported basis functions each spanning at most two adjacent cells, which makes the computation of the local stiffness matrices and the assembling of the global stiffness matrix implementable by following the standard procedure. Some numerical experiments are given to show the implementability and the performance of the new kind of spaces. The cooperation of conforming and nonconforming finite element spaces leads to new discretization schemes of the Hodge Laplace problem.

Keywords: exterior differential form; nonconforming finite element space; discrete Poincaré inequality; discrete de Rham complex; commutative diagram; discrete Helmholtz-Hodge decomposition; discrete Poincaré-Lefschetz duality; Hodge Laplace problem; nonconforming finite element exterior calculus.

1. Introduction

Conforming finite element exterior calculus has been extensively studied and well established based on conforming finite elements for exterior differential forms; we refer to, e.g., Arnold et al. (2010); Arnold (2018); Arnold et al. (2006); Boffi et al. (2013); Hiptmair (2002) and the references therein for details. Naturally, the research has now reached a point where extension is appropriate to nonconforming methods. Actually, for some specific applications, such as the $H\Lambda^k \cap H^*\Lambda^k$ problems, in general people can not establish reasonable conforming primal finite element spaces, and we are led to the cruciality of investigating the nonconforming discretizations to exterior differential operators and forms. Meanwhile, well-designed nonconforming methods, with the (lowest-degree) Crouzeix-Raviart element (Crouzeix & Raviart, 1973) being a typical example, can possess many characteristics that conforming ones lack, including, e.g.,

- Different from conforming interpolators discussed before (Christiansen & Winther, 2008; Clément, 1975; Gawlik et al., 2021; Licht, 2019b,a; Ern & Guermond, 2017; Scott & Zhang, 1990; Falk & Winther, 2014), the Crouzeix-Raviart element admits a cell-wise defined[1] stable interpolator which works for functions in H^1 without using the inter-cell regularization, smoothing or averaging techniques.
- In the construction of Helmholtz orthogonal decomposition of piecewise constants, which cannot be established when restricted to conforming element spaces, the lowest-degree Crouzeix-Raviart element plays an irreplaceable role (Arnold & Falk, 1989; Monk, 1991).
- Applied to the computation of Laplacian eigenvalues, the lowest-degree Crouzeix-Raviart element scheme may yield asymptotic lower bounds to the exact eigenvalues (Armentano & Durán, 2004), which differs essentially from conforming ones.

These properties may indicate the potential theoretical and practical significance of nonconforming methods compared to conforming ones. Some specific nonconforming complexes, such as the well-known 2-D discrete Stokes complex formulated by the Morley element, the Crouziex-Raviart element and piecewise constant (Falk & Morley, 1990), have been established, though, a general construction for the nonconforming Hilbert complex of differential forms, namely one that connects nonconforming spaces for $H\Lambda^k$ in general $\mathbb{R}^n$, seems still absent. This paper will hence investigate nonconforming finite element spaces for general exterior differential forms, by particularly generalizing the Crouzeix-Raviart element for $H\Lambda^0$ to a unified family for $H\Lambda^k$ for $0 \leqslant k \leqslant n$ in $\mathbb{R}^n$ by a novel approach, and systematic theory of nonconforming finite element exterior calculus will then be established based on these spaces.

Attempts to generalize the Crouzeix-Raviart elements have been devoted before to the $H(\mathrm{div})$ problems (Arbogast & Correa, 2016; Shi & Pei, 2008; Quan et al., 2022). Following directly from Crouzeix-Raviart element, these elements all use the integral of the normal components as nodal parameters. For these elements, the crucial property of the Crouzeix-Raviart element, namely completely cell-wise defined nodal interpolator, cannot be validated for functions with only $H(\mathrm{div})$ regularity, nor can an associated discrete Helmholtz decomposition be established. Further, if we try to embed such an $H(\mathrm{div})$ element into a discretized de Rham complex, which is a crucial issue for the discretization of exterior differential operators, the continuity restriction for the corresponding H^1 finite element is the evaluation at vertices. As well known, the continuity of the evaluation at vertices is neither sufficient nor necessary for a finite element to work for H^1 problems, and the weak continuity condition for these $H(\mathrm{div})$ elements is not as reasonable as the original Crouzeix-Raviart element. It is suggested in Bringmann et al. (2024) that vector Crouzeix-Raviart element can be used for $H(\mathrm{curl})$ in three dimension; though, the same obstacles can be come across.

Inspired by a new interpretation to the Crouzeix-Raviart element, instead of imposing local continuity primally, a novel approach given in this paper of establishing the space is to reveal and mimic the relationship between adjoint operators. Actually, beyond being a **consequence**, the well-known integration by part formula which reads, on the lowest-degree Crouzeix-Raviart element space V_h^{CR} and the lowest-degree Raviart-Thomas element space $\underset{\sim}{V}_{h0}^{\mathrm{RT}}$ on a grid $\mathcal{G}_h$,

$$\sum_{T\in\mathcal{G}_h}\int_T \nabla v_h \underset{\sim}{\tau}_h + \int_T v_h \mathrm{div} \underset{\sim}{\tau}_h = 0, \quad \text{for } v_h \in V_h^{\mathrm{CR}} \text{ and } \underset{\sim}{\tau}_h \in \underset{\sim}{V}_{h0}^{\mathrm{RT}} \tag{1.1}$$

[1] Here and in the sequel, by "locally defined" or "cell-wise defined", we mean if two functions u and v are equal on a cell T, then their respective interpolations $\mathbb{I}u$ and $\mathbb{I}v$ are equal on T.

also serves as a **sufficient** condition for a piecewise linear polynomial function to belong to $V_h^{\rm CR}$. Namely, $V_h^{\rm CR}$ can be equivalently figured out as

$$V_h^{\rm CR} = \left\{ v_h \text{ is piecewise linear, such that } \sum_{T\in\mathcal{G}_h} \int_T \nabla v_h \underset{\sim}{\tau}_h + \int_T v_h {\rm div} \underset{\sim}{\tau}_h = 0\ \forall\, \underset{\sim}{\tau}_h \in \underset{\sim}{V}_{h0}^{\rm RT} \right\}. \qquad (1.2)$$

This way the adjoint relation between $({\rm div}, H_0({\rm div}))$ and (∇, H^1) is mimicked. This observation hints quite a natural approach to construct a finite element space by constructing discrete adjoint relationships, and is applied for spaces $H\Lambda^k$ in this paper.

By the aid of the existing conforming finite element spaces $\mathbf{W}_h^*\Lambda^k$ for $H^*\Lambda^k$ by piecewise Whitney forms, in this paper, a family of nonconforming finite element spaces for $H\Lambda^k$ is constructed by mimicking the dual connections between adjoint operators $\mathbf{d}$ and δ. Advantages emerge naturally from the construction of the finite element spaces. A first one is that the consistency error can be controlled by the approximation of the adjoint conforming spaces. Then, cell-wise defined global interpolators can be constructed for functions in $H\Lambda^k$ without extra regularity; the interpolators are stable in broken $H\Lambda^k$ norm and provide optimal approximation to functions in $H\Lambda^k$. Combined with the global interpolators, these newly constructed spaces are connected by piecewise operations of $\mathbf{d}^k$ to form nonconforming finite element de Rham complexes, as well as commutative diagrams with the de Rham Hilbert complexes. Further, the Helmholtz and Hodge decompositions of the piecewise constant k-forms follow from the discrete adjoint relation. It is worth noting that the Poincaré-Leftschetz duality can be reconstructed as Theorem 28 by the respective discrete harmonic spaces by conforming and nonconforming finite element spaces, where the space and its dual are identical, which differs from previous constructions with two individual discrete spaces that can be asymptotically made arbitrarily close. With these structural properties given in Section 3, a framework of nonconforming finite element exterior calculus is established, and is naturally linked to the classical conforming one by the discrete complex duality (3.7) and the discrete Poincaré-Lefschetz duality.

Since nonconforming finite element spaces are constructed for $(\mathbf{d}^k, H\Lambda^k)$ and particularly discrete Hodge decompositions are constructed accordingly, new discretization schemes can be developed. Meanwhile, dual structures can be further investigated with more applications. We investigate the dual roles of conforming and nonconforming spaces by constructing some new finite element schemes for the Hodge Laplace problem with nonconforming spaces. The two finite element spaces connect with each other within their respective discretization schemes through classical mixed formulations, and their roles are complementary within the discretization scheme of a new mixed formulation.

On the other hand, in contrast to the conforming Whitney forms, the nonconforming finite element spaces defined in this paper may not correspond to a “finite element”(triple) in Ciarlet’s sense (Ciarlet, 1978). Therefore, some basic features of the finite element methods cannot be dealt with in standard ways. Two main obstacles are: (1), it is not any longer straightforward to figure out the basis functions of the global finite element spaces, and (2), it is difficult, if not impossible, to follow the standard procedure to prove the uniform discrete Poincaré inequalities. In this paper, we develop nonstandard approaches to circumvent the obstacles. For every newly designed finite element space, we prove the existence of a set of basis functions which each is supported on no more than two cells, and the relevant numerical scheme can be implemented by the standard routine for the finite element in Ciarlet’s sense. Some numerical experiments are provided to verify the implementability of the new finite element functions. As a discrete analogue to the closed range theorem (see Section B for a quantifiable formulation), we prove that the constant of the discrete Poincaré inequality of a newly designed finite element space is asymptotically equal to that of an associated conforming Whitney form space which has been proved uniformly bounded; it follows that the discrete Poincaré inequality holds uniformly for the new spaces.

Finally we remark that the reciprocal causation (3.7) between two discrete complexes can be viewed a discrete analogue of the complex duality composed by adjoint operator pairs (namely $\mathbf{d}$ and δ), defined in Arnold (2018, Section 4.1.2). We note that kinds of dual complexes in their respective specific senses used to be studied in, e.g., Arnold et al. (2009); Berchenko-Kogan (2021); Buffa & Christiansen (2007); Dłotko & Specogna (2013); Jain et al. (2021); Nakata et al. (2019); Oden (1972); Schöberl (2008); Wieners & Wohlmuth (2011) and Licht (2017). Prior works primarily address representations in the dual spaces, while the constructions in the present paper focus on the spaces of finite element functions. We particularly remark that, such construction of finite element spaces and associated complexes in the present paper by mimicking the dual connections between adjoint operators $\mathbf{d}$ and δ, cf. (1.2) above and (3.1) and (3.2) below, is relevant to but different from the complexes of discrete distributional differential forms (Braess & Schöberl, 2008; Licht, 2017; Christiansen & Licht, 2020; Hu et al., 2025). Actually, at the same time as nonconforming finite element spaces $\mathbf{W}_h^{\rm nc}\Lambda^k(\mathbf{W}_{h0}^{\rm nc}\Lambda^k)$, without or with homogeneous boundary restrictions, can be constructed as discrete adjoint spaces of the conforming finite element spaces $\mathbf{W}_{h0}^*\Lambda^k(\mathbf{W}_h^*\Lambda^k)$, the spaces $\mathbf{W}_h^*\Lambda^k$ can also be figured out by the discrete adjoint relationship to $\mathbf{W}_h^{\rm nc}\Lambda^k$; see Remarks 1 and 17; a duality between two complexes by finite element spaces respectively fit for $H\Lambda^k$ and $H_0^*\Lambda^k$ is established in this paper. Note that both conforming and nonconforming known spaces can be chosen as accompanying spaces; this freedom further brings convenience to the construction of finite element spaces, and is especially crucial for the primal discretization to the Hodge-Laplace problem, which will be investigated in future.

The remainder of the paper is organized as follows. In the remaining part of this section, we collect some preliminaries and notations. In Section 2, we use the two-dimensional $H(\mathrm{div})$ problem for instance to illustrate the main features of the new type of finite element spaces, including the construction of the new space, the locally-supported basis functions, the basic error estimation by cell-wise defined interpolators, and numerical verifications for the validity of the new finite element spaces. In Section 3, a family of nonconforming finite element spaces are constructed for $H\Lambda^k$ in $\mathbb{R}^n$, $0 \leqslant k \leqslant n$, with the Crouzeix-Raviart element space being the one for $H\Lambda^0$. Based on these finite element spaces, theory of nonconforming finite element exterior calculus is constructed, including the Helmholtz/Hodge decomposition for piecewise constant k-forms, the discrete Poincaré-Lefschetz duality, the discrete de Rham complex and commutative diagrams. Then in Section 4, the newly-designed nonconforming spaces are used for the discretization of the Hodge Laplace problem. The correspondent and complementary connections between the conforming and nonconforming spaces are investigated with classical and new mixed formulations. Finally, in Section 5, some conclusions and discussions are given.

Preliminaries and Notations In the sequel of the paper, we use $\mathscr{N}$ and $\mathscr{R}$ to denote the **N**ull space and the **R**ange of certain operators. Namely, for example, $\mathscr{N}(\mathbf{T},\mathbf{D})$ denotes $\{\mathbf{v}\in\mathbf{D}:\mathbf{T}\mathbf{v}=0\}$, and $\mathscr{R}(\mathbf{T},\mathbf{D})$ denotes $\{\mathbf{T}\mathbf{v}:\mathbf{v}\in\mathbf{D}\}$. For a Hilbert space $\mathbf{H}$, we use the notations $\oplus_{\mathbf{H}}^{\perp}$ and $\ominus_{\mathbf{H}}^{\perp}$ to denote the orthogonal summation and orthogonal difference; namely, for two spaces $\mathbf{A}$ and $\mathbf{B}$ in $\mathbf{H}$, the presentation $\mathbf{A}\oplus_{\mathbf{H}}^{\perp}\mathbf{B}$ implies that $\mathbf{A}$ and $\mathbf{B}$ are orthogonal in $\mathbf{H}$, and evaluates as the direct summation of $\mathbf{A}$ and $\mathbf{B}$;

for $\mathbf{A}\subset\mathbf{B}\subset\mathbf{H}$, $\mathbf{B}\ominus^{\perp}_{\mathbf{H}}\mathbf{A}$ evaluates as the orthogonal complementation of $\mathbf{A}$ in $\mathbf{B}$. The subscript $\mathbf{H}$ can occasionally be dropped.

For Ω a domain and $T\subset\Omega$, we use $E_T^\Omega: L^1(T)\to L^1(\Omega)$ for the extension operator defined by $E_T^\Omega v=v$ on T and $E_T^\Omega v=0$ elsewhere. For $V_T\subset L^1(T)$, we use $E_T^\Omega V_T$ for short of $\mathscr{R}(E_T^\Omega,V_T)$.

We use $\mathbf{d}^k$ and δ_k for the exterior *differential* and *codifferential* operators on Λ^k. $\delta_k=(-1)^{kn}\star\mathbf{d}^{n-k}\star$, $\star$ being the Hodge star operator. Denote, on the domain Ξ,

$$H\Lambda^k(\Xi):=\left\{\omega\in L^2\Lambda^k(\Xi):\mathbf{d}^k\omega\in L^2\Lambda^{k+1}(\Xi)\right\},\quad 0\leqslant k\leqslant n-1,$$

and by $H_0\Lambda^k(\Xi)$ the closure of $\mathscr{C}_0^\infty\Lambda^k(\Xi)$ in $H\Lambda^k(\Xi)$. Denote

$$H^*\Lambda^k(\Xi):=\left\{\mu\in L^2\Lambda^k(\Xi):\delta_k\mu\in L^2\Lambda^{k-1}(\Xi)\right\},\quad 1\leqslant k\leqslant n,$$

and $H_0^*\Lambda^k(\Xi)$ the closure of $\mathscr{C}_0^\infty\Lambda^k(\Xi)$ in $H^*\Lambda^k(\Xi)$. Ξ can occasionally be dropped. The spaces of harmonic forms are $\mathfrak{H}\Lambda^k:=\mathscr{N}(\mathbf{d}^k,H\Lambda^k)\ominus^{\perp}\mathscr{R}(\mathbf{d}^{k-1},H\Lambda^{k-1})$, $\mathfrak{H}_0\Lambda^k:=\mathscr{N}(\mathbf{d}^k,H_0\Lambda^k)\ominus^{\perp}\mathscr{R}(\mathbf{d}^{k-1},H_0\Lambda^{k-1})$, $\mathfrak{H}^*\Lambda^k:=\mathscr{N}(\delta_k,H^*\Lambda^k)\ominus^{\perp}\mathscr{R}(\delta_{k+1},H^*\Lambda^{k+1})$, and $\mathfrak{H}_0^*\Lambda^k:=\mathscr{N}(\delta_k,H_0^*\Lambda^k)\ominus^{\perp}\mathscr{R}(\delta_{k+1},H_0^*\Lambda^{k+1})$. As the Helmholtz decompositions hold that

$$\mathscr{N}(\mathbf{d}^k,H\Lambda^k)\oplus^{\perp}\mathscr{R}(\delta_{k+1},H_0^*\Lambda^{k+1})=L^2\Lambda^k=\mathscr{R}(\mathbf{d}^{k-1},H\Lambda^{k-1})\oplus^{\perp}\mathscr{N}(\delta_k,H_0^*\Lambda^k),$$

it follows that $\mathfrak{H}\Lambda^k=\mathfrak{H}_0^*\Lambda^k$ and $\mathfrak{H}_0\Lambda^k=\mathfrak{H}^*\Lambda^k$. This is the Poincaré-Lefschetz duality(cf. Arnold (2018, Section 4.5.5)) which links two dual complexes connected by $\mathbf{d}^k$ and δ_k, respectively.

The space of Whitney forms is denoted as (Arnold et al., 2006, 2010; Arnold, 2018) $\mathscr{P}_1^-\Lambda^k=\mathscr{P}_0\Lambda^k+\kappa(\mathscr{P}_0\Lambda^{k+1})$, where the Koszul operator κ is $\kappa(\mathrm{d}x^{\alpha_1}\wedge\cdots\wedge\mathrm{d}x^{\alpha_k}):=\sum_{j=1}^{k}(-1)^{j+1}x^{\alpha_j}\,\mathrm{d}x^{\alpha_1}\wedge\cdots\wedge\mathrm{d}x^{\alpha_{j-1}}\wedge\mathrm{d}x^{\alpha_{j+1}}\wedge\cdots\wedge\mathrm{d}x^{\alpha_k}$ for

$$\alpha\in\mathbb{IX}_{k,n}:=\left\{\alpha=(\alpha_1,\dots,\alpha_k)\in\mathbb{N}^k:1\leqslant\alpha_1<\alpha_2<\cdots<\alpha_k\leqslant n,\ \mathbb{N}\text{ the set of integers}\right\},$$

the set of k-indices, $k\leqslant n$. Note that $\mathscr{P}_1^-\Lambda^0=\mathscr{P}_1\Lambda^0$ and $\mathscr{P}_1^-\Lambda^n=\mathscr{P}_0\Lambda^n$. Denote the Whitney forms associated with the operator δ_k by $\mathscr{P}_1^{*,-}\Lambda^k:=\star(\mathscr{P}_1^-\Lambda^{n-k})$. Note that

$$\mathscr{N}(\mathbf{d}^k,\mathscr{P}_1^-\Lambda^k)=\mathscr{R}(\mathbf{d}^{k-1},\mathscr{P}_1^-\Lambda^{k-1})=\mathscr{P}_0\Lambda^k=\mathscr{R}(\delta_{k+1},\mathscr{P}_1^{*,-}\Lambda^{k+1})=\mathscr{N}(\delta_k,\mathscr{P}_1^{*,-}\Lambda^k).\tag{1.3}$$

Denote, on a simplicial subdivision $\mathscr{G}_h$ of Ω, $0\leqslant k\leqslant n$,

$$\mathscr{P}_1^-\Lambda^k(\mathscr{G}_h):=\bigoplus_{T\in\mathscr{G}_h}E_T^\Omega\mathscr{P}_1^-\Lambda^k(T),\ \text{and}\ \mathscr{P}_1^{*,-}\Lambda^k(\mathscr{G}_h):=\bigoplus_{T\in\mathscr{G}_h}E_T^\Omega\mathscr{P}_1^{*,-}\Lambda^k(T).\tag{1.4}$$

Here and in the sequel, the subscript "$\cdot_h$" denotes mesh dependence. In particular, an operator with the subscript "$\cdot_h$" indicates that the operation is performed cell by cell.

The conforming finite element spaces with Whitney forms are $\mathbf{W}_h\Lambda^k:=\mathscr{P}_1^-(\mathscr{G}_h)\cap H\Lambda^k$, $\mathbf{W}_{h0}\Lambda^k:=\mathscr{P}_1^-(\mathscr{G}_h)\cap H_0\Lambda^k$, $\mathbf{W}_h^*\Lambda^k:=\mathscr{P}_1^{*,-}(\mathscr{G}_h)\cap H^*\Lambda^k$, and $\mathbf{W}_{h0}^*\Lambda^k:=\mathscr{P}_1^{*,-}(\mathscr{G}_h)\cap H_0^*\Lambda^k$. Note that the spaces defined this way are respectively identical to the finite element spaces with piecewise Whitney forms defined by the continuity of the nodal parameters (Arnold, 2018). Denote the spaces of discrete harmonic forms by $\mathfrak{H}_h\Lambda^k := \mathcal{N}(\mathbf{d}^k, \mathbf{W}_h\Lambda^k) \ominus^\perp \mathcal{R}(\mathbf{d}^{k-1}, \mathbf{W}_h\Lambda^{k-1})$, $\mathfrak{H}_{h0}\Lambda^k := \mathcal{N}(\mathbf{d}^k, \mathbf{W}_{h0}\Lambda^k) \ominus^\perp \mathcal{R}(\mathbf{d}^{k-1}, \mathbf{W}_{h0}\Lambda^{k-1})$, $\mathfrak{H}^*_h\Lambda^k := \mathcal{N}(\delta_k, \mathbf{W}^*_h\Lambda^k) \ominus^\perp \mathcal{R}(\delta_{k+1}, \mathbf{W}^*_h\Lambda^{k+1})$, and $\mathfrak{H}^*_{h0}\Lambda^k := \mathcal{N}(\delta_k, \mathbf{W}^*_{h0}\Lambda^k) \ominus^\perp \mathcal{R}(\delta_{k+1}, \mathbf{W}^*_{h0}\Lambda^{k+1})$.

Given T a simplex, denote, associated with T, $\tilde{x}^j = x^j - c_j$, where c_j is a constant such that $\int_T \tilde{x}^j = 0$, and κ_T, a Koszul operator on T, by for $\alpha \in \mathbb{IX}_{k,n}$

$$\kappa_T(\mathrm{dx}^{\alpha_1} \wedge \cdots \wedge \mathrm{dx}^{\alpha_k}) := \sum_{j=1}^{k} (-1)^{(j+1)} \tilde{x}^{\alpha_j} \mathrm{dx}^{\alpha_1} \wedge \ldots \mathrm{dx}^{\alpha_{j-1}} \wedge \mathrm{dx}^{\alpha_{j+1}} \wedge \cdots \wedge \mathrm{dx}^{\alpha_k}.$$

Then $\mathbf{d}^{k-1}\kappa_T(\mathrm{dx}^{\alpha_1} \wedge \ldots \mathrm{dx}^{\alpha_k}) = k\,\mathrm{dx}^{\alpha_1} \wedge \ldots \mathrm{dx}^{\alpha_k}$. By the aid of κ_T, we can rewrite the Whitney forms as $\mathcal{P}_1^-\Lambda^k(T) = \mathcal{P}_0\Lambda^k(T) \oplus^\perp \kappa_T(\mathcal{P}_0\Lambda^{k+1}(T))$, orthogonal in $L^2\Lambda^k(T)$. We further use κ_h to denote the operation of κ_T cell by cell. Denote $\kappa^\delta := \star \circ \kappa \circ \star$, $\kappa_T^\delta := \star \circ \kappa_T \circ \star$, and $\kappa_h^\delta := \star \circ \kappa_h \circ \star$.

2. A nonconforming $H(\mathrm{div})$ finite element space

In this section, we use the two-dimensional $H(\mathrm{div})$ problem for instance to illustrate the main features of the new type of finite element spaces studied in this paper.

Let $\Omega \subset \mathbb{R}^2$ denote a polygon. As usual, we use ∇ and div to denote the gradient operator and divergence operator, respectively, and we use $H^1(\Omega)$, $H^1_0(\Omega)$, $H(\mathrm{div}, \Omega)$, $H_0(\mathrm{div}, \Omega)$, $L^2(\Omega)$ and $L^2_0(\Omega)$ to denote certain Sobolev (Lebesgue) spaces. For here, we denote vector-valued quantities by undertilde "$\undertilde{\cdot}$". We use $(\cdot,\cdot)$ with subscripts to represent L^2 inner product.

For this planar domain, we specifically use $\mathcal{T}_h$ for a shape-regular subdivision of Ω with mesh size h that consists of triangles, such that $\overline{\Omega} = \cup_{T\in\mathcal{T}_h}\overline{T}$ and every boundary vertex is connected to at least one interior vertex. Denote by $\mathcal{E}_h$, $\mathcal{E}_h^i$, $\mathcal{E}_h^b$, $\mathcal{X}_h$, $\mathcal{X}_h^i$ and $\mathcal{X}_h^b$ the set of edges, interior edges, boundary edges, vertices, interior vertices and boundary vertices, respectively. We use $\mathbf{n}$ for the outward unit normal vector with respect to a triangle.

Let $\mathbb{V}_h^1$ denote the continuous piecewise linear element space, and $\undertilde{V}_h^{\mathrm{RT}}$ denote the classical Raviart-Thomas element space (Raviart & Thomas, 1977) of lowest degree on $\mathcal{T}_h$. Denote $\mathbb{V}^1_{h0} := \mathbb{V}^1_h \cap H^1_0(\Omega)$ and $\undertilde{V}_{h0}^{\mathrm{RT}} := \undertilde{V}_h^{\mathrm{RT}} \cap H_0(\mathrm{div}, \Omega)$. On a triangle T, denote the space of the lowest-degree Raviart-Thomas shape functions by $\mathbb{RT}(T) := \mathrm{span}\{\undertilde{\alpha} + \beta\undertilde{x} : \undertilde{\alpha} \in \mathbb{R}^2, \beta \in \mathbb{R}\}$. Then

$$\mathcal{R}(\mathrm{div}, \mathbb{RT}(T)) = \mathbb{R} = \mathcal{N}(\nabla, P_1(T)), \quad \text{and} \quad \mathcal{N}(\mathrm{div}, \mathbb{RT}(T)) = \mathbb{R}^2 = \mathcal{R}(\nabla, P_1(T)). \tag{2.1}$$

Denote $\mathbb{RT}(\mathcal{T}_h) := \bigoplus_{T\in\mathcal{T}_h} E_T^\Omega \mathbb{RT}(T)$. We define the nonconforming finite element spaces

$$\mathbb{RT}_h^{\mathrm{nc}} := \left\{ \undertilde{\tau}_h \in \mathbb{RT}(\mathcal{T}_h) : \sum_{T\in\mathcal{T}_h} (\undertilde{\tau}_h, \nabla \mathbf{v}_h)_T + (\mathrm{div}\undertilde{\tau}_h, \mathbf{v}_h)_T = 0,\ \forall \mathbf{v}_h \in \mathbb{V}^1_{h0} \right\}, \tag{2.2}$$

and

$$\mathbb{RT}_{h0}^{\mathrm{nc}} := \left\{ \undertilde{\tau}_h \in \mathbb{RT}(\mathcal{T}_h) : \sum_{T\in\mathcal{T}_h} (\undertilde{\tau}_h, \nabla \mathbf{v}_h)_T + (\mathrm{div}\undertilde{\tau}_h, \mathbf{v}_h)_T = 0,\ \forall \mathbf{v}_h \in \mathbb{V}^1_{h} \right\}. \tag{2.3}$$

Note that $\mathbb{RT}_h^{\mathrm{nc}}$ does not confirm to Ciarlet's finite element definition. In Section 2.1, we will present sets of locally supported basis functions for each of $\mathbb{RT}_h^{\mathrm{nc}}$ and $\mathbb{RT}_{h0}^{\mathrm{nc}}$ for their implementability. In

Section 2.2, we establish a cell-wise defined projective interpolator for $H(\mathrm{div})$, and prove optimal approximation and stability properties of $\mathbb{RT}_h^{\rm nc}$ and $\mathbb{RT}_{h0}^{\rm nc}$ directly without the aid of the classical Raviart-Thomas element.

Remark 1 *Evidently,*

$$\mathbb{V}_{h0}^1=\left\{\mathbf{v}_h \textit{ is piecewise linear, such that } \sum_{T\in\mathcal{T}_h}(\underset{\sim}{\tau}_h,\nabla\mathbf{v}_h)_T+(\mathrm{div}\underset{\sim}{\tau}_h,\mathbf{v}_h)_T=0,\ \forall\underset{\sim}{\tau}_h\in\mathbb{RT}_h^{\rm nc}\right\}, \tag{2.4}$$

and

$$\mathbb{V}_{h}^1=\left\{\mathbf{v}_h \textit{ is piecewise linear, such that } \sum_{T\in\mathcal{T}_h}(\underset{\sim}{\tau}_h,\nabla\mathbf{v}_h)_T+(\mathrm{div}\underset{\sim}{\tau}_h,\mathbf{v}_h)_T=0,\ \forall\mathbf{v}_h\in\mathbb{RT}_{h0}^{\rm nc}\right\}. \tag{2.5}$$

2.1. *Locally supported global basis functions of $\mathbb{RT}_{h0}^{\rm nc}$ and $\mathbb{RT}_h^{\rm nc}$*

2.1.1. Structures of $\mathbb{RT}(T)$ on a triangle T and $\mathbb{RT}(\mathcal{T}_h)$ on $\mathcal{T}_h$

For a cell $T\in\mathcal{T}_h$, we use a_i (located at $\underset{\sim}{a}_i$) and e_i for the vertices and opposite edges, h_i being the height on e_i, $i=1:3$. Let λ_i be the barycentric coordinates. Let $|e_i|$ and $|h_i|$ denote the length of e_i and h_i, respectively, and let S denote the area; cf. Figure 1.

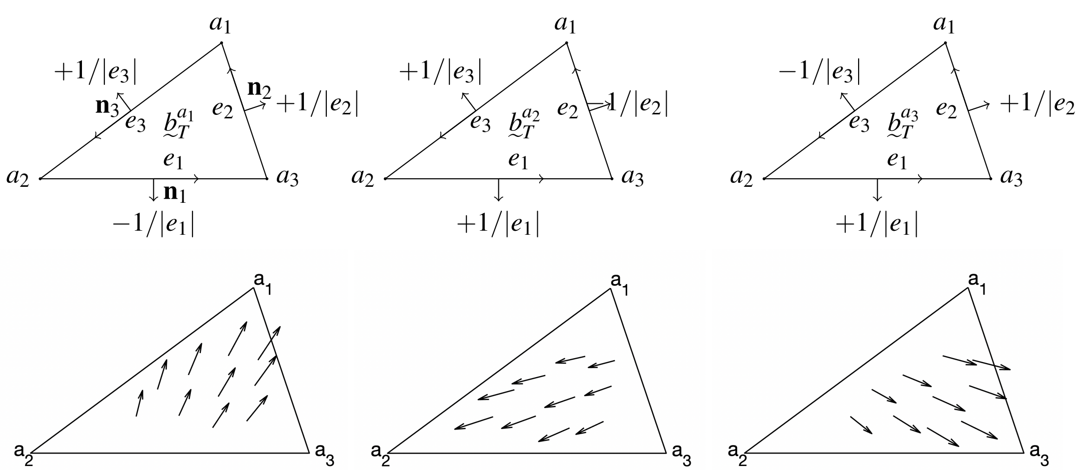

FIG. 1. Illustration of the fields of the three basis functions of $\mathbb{RT}(T)$ on a cell T (bottom row). We pay particular attention to the sign of the outward normal component at every edge (top row).

Denote

$$\underset{\sim}{b}_T^{a_i}:=\frac{1}{2S}(\underset{\sim}{x}+\underset{\sim}{a}_i-\underset{\sim}{a}_j-\underset{\sim}{a}_k),\ \ i=1,2,3,\ \{i,j,k\}=\{1,2,3\}. \tag{2.6}$$

Then, $\left\{\underset{\sim}{b}_T^{a_i}, i=1,2,3\right\}$ form a basis of $\mathbb{RT}(T)$. Particularly, $\underset{\sim}{b}_T^{a_i}\cdot\mathbf{n}_j|_{e_j}=(1-2\delta_{ij})/|e_j|$, and

$$(\underset{\sim}{b}_T^{a_i},\nabla\lambda_j)_T+(\mathrm{div}\underset{\sim}{b}_T^{a_i},\lambda_j)_T=\delta_{ij},\ \ 1\leqslant i,j\leqslant 3. \tag{2.7}$$

The identities (2.1) confirm the existence of a basis of $\mathbb{RT}(T)$ that satisfies the dual relation (2.7), and (2.6) further gives the precise formulation of them. See Figure 1 for the illustrations and profiles of the local basis functions. Then

$$\mathbb{RT}(\mathscr{T}_h)=\bigoplus_{T\in\mathscr{T}_h}E_T^\Omega\mathbb{RT}(T)=\bigoplus_{T\in\mathscr{T}_h}\bigoplus_{M\in\mathscr{X}_h\cap\partial T}\mathrm{span}\left\{E_T^\Omega \underset{\sim}{b}_T^M\right\}=\bigoplus_{M\in\mathscr{X}_h}\bigoplus_{\partial T\ni M}\mathrm{span}\left\{E_T^\Omega \underset{\sim}{b}_T^M\right\}.$$

2.1.2. Two types of basis functions in $\mathbb{RT}_{h0}^{\rm nc}$ and $\mathbb{RT}_h^{\rm nc}$
For $M\in\mathscr{X}_h$, denote by ψ_M the basis function of $\mathbb{V}_h^1$ such that $\psi_M(M)=1$ and ψ_M vanishes on other vertices. We can rewrite (2.7) to the lemma below.

Lemma 2 *For $M,M'\in\mathscr{X}_h$ and $T,T'\in\mathscr{T}_h$, such that $M\in\partial T$ and $M'\in\partial T'$, with*

$$\delta_{MM'}\ \textit{denoting}\ \left\{\begin{array}{ll}1, & M=M'\\ 0, & M\neq M'\end{array}\right.\quad \textit{and}\quad \delta_{TT'}\ \textit{denoting}\ \left\{\begin{array}{ll}1, & T=T'\\ 0, & T\neq T'\end{array}\right.,$$

it holds that

$$(E_T^\Omega \underset{\sim}{b}_T^M,\nabla\psi_{M'})_{T'}+(\mathrm{div}E_T^\Omega \underset{\sim}{b}_T^M,\psi_{M'})_{T'}=\delta_{MM'}\delta_{TT'}.$$

Denote, for $M\in\mathscr{X}_h$,

$$\mathscr{B}_M:=\left\{\underset{\sim}{\tau}_h\in\bigoplus_{\partial T\ni M}\mathrm{span}\left\{E_T^\Omega \underset{\sim}{b}_T^M\right\}:\sum_{T\in\mathscr{T}_h}(\underset{\sim}{\tau}_h,\nabla\psi_M)_T+(\mathrm{div}\underset{\sim}{\tau}_h,\psi_M)_T=0\right\},\tag{2.8}$$

and

$$\mathscr{C}_M:=\bigoplus_{\partial T\ni M}\mathrm{span}\left\{E_T^\Omega \underset{\sim}{b}_T^M\right\}.\tag{2.9}$$

Then $\mathscr{B}_M\subset\mathscr{C}_M$. We present the structures of $\mathbb{RT}_{h0}^{\rm nc}$ and $\mathbb{RT}_h^{\rm nc}$ in the lemma below.

Lemma 3 1. *If $M\neq N\in\mathscr{X}_h$, $\mathscr{C}_M\cap\mathscr{C}_N=\{0\}$;*
2. $\mathbb{RT}_{h0}^{\rm nc}=\bigoplus_{M\in\mathscr{X}_h}\mathscr{B}_M$;
3. $\mathbb{RT}_h^{\rm nc}=\left[\bigoplus_{M\in\mathscr{X}_h^b}\mathscr{C}_M\right]\oplus\left[\bigoplus_{M\in\mathscr{X}_h^i}\mathscr{B}_M\right].$

Proof The first item follows directly by definition. For the second, by (2.3) and Lemma 2,

$$\mathbb{RT}_{h0}^{\rm nc}=\left\{\underset{\sim}{\tau}_h\in\bigoplus_{M\in\mathscr{X}_h}\bigoplus_{\partial T\ni M}\mathrm{span}\left\{E_T^\Omega \underset{\sim}{b}_T^M\right\}:\sum_{T\in\mathscr{T}_h}(\underset{\sim}{\tau}_h,\nabla\psi_N)_T+(\mathrm{div}\underset{\sim}{\tau}_h,\psi_N)_T=0,\ \forall N\in\mathscr{X}_h\right\}$$
$$=\bigoplus_{M\in\mathscr{X}_h}\left\{\underset{\sim}{\tau}_h\in\bigoplus_{\partial T\ni M}\mathrm{span}\left\{E_T^\Omega \underset{\sim}{b}_T^M\right\}:\sum_{T\in\mathscr{T}_h,\partial T\ni M}(\underset{\sim}{\tau}_h,\nabla\psi_M)_T+(\mathrm{div}\underset{\sim}{\tau}_h,\psi_M)_T=0\right\},$$

and the second item follows. For the third,

$$
\begin{aligned}
\mathbb{RT}_h^{\rm nc} &= \left\{ \underset{\sim}{\tau}_h \in \bigoplus_{M\in\mathscr{X}_h} \bigoplus_{\partial T\ni M} \mathrm{span}\left\{E_T^{\Omega}\underset{\sim}{b}_T^M\right\} : \sum_{T\in\mathscr{T}_h} (\underset{\sim}{\tau}_h, \nabla\psi_N)_T + (\mathrm{div}\underset{\sim}{\tau}_h, \psi_N)_T = 0,\ \forall N\in\mathscr{X}_{h0} \right\} \\
&= \left\{ \underset{\sim}{\tau}_h \in \bigoplus_{M\in\mathscr{X}_h^i} \bigoplus_{\partial T\ni M} \mathrm{span}\left\{E_T^{\Omega}\underset{\sim}{b}_T^M\right\} : \sum_{T\in\mathscr{T}_h} (\underset{\sim}{\tau}_h, \nabla\psi_N)_T + (\mathrm{div}\underset{\sim}{\tau}_h, \psi_N)_T = 0,\ \forall N\in\mathscr{X}_{h0} \right\} \\
&\qquad \bigoplus \left\{ \underset{\sim}{\tau}_h \in \bigoplus_{M\in\mathscr{X}_h^b} \bigoplus_{\partial T\ni M} \mathrm{span}\left\{E_T^{\Omega}\underset{\sim}{b}_T^M\right\} \right\} \\
&= \left[\bigoplus_{M\in\mathscr{X}_h^i} \left\{ \underset{\sim}{\tau}_h \in \bigoplus_{\partial T\ni M} \mathrm{span}\left\{E_T^{\Omega}\underset{\sim}{b}_T^M\right\} : \sum_{T\in\mathscr{T}_h} (\underset{\sim}{\tau}_h, \nabla\psi_M)_T + (\mathrm{div}\underset{\sim}{\tau}_h, \psi_M)_T = 0 \right\} \right] \\
&\qquad \bigoplus \left[\bigoplus_{M\in\mathscr{X}_h^b} \bigoplus_{\partial T\ni M} \mathrm{span}\left\{E_T^{\Omega}\underset{\sim}{b}_T^M\right\} \right].
\end{aligned}
$$

The third item follows. This completes the proof. □

2.1.3. Profiles of $\mathscr{B}_M$ and $\mathscr{C}_M$

Lemma 4 *Given a vertex M that is shared by m triangles, $\dim(\mathscr{B}_M) = m-1$. There exist a set of basis functions of $\mathscr{B}_M$, each of which is supported on no more than two cells.*

Proof The support of ψ_M consists of m triangles. Denote by T_i, $1\leqslant i\leqslant m$, the m triangles that share M. The basis functions in $\mathscr{B}_M$ then take the form $\sum_{i=1}^m \gamma_i \underset{\sim}{b}_{T_i}^M$, satisfying

$$
\sum_{i=1}^m \left[(\gamma_i \underset{\sim}{b}_{T_i}^M, \nabla(\psi_M|_{T_i}))_{T_i} + (\gamma_i \mathrm{div}\underset{\sim}{b}_{T_i}^M, \psi_M|_{T_i})_{T_i} \right] = 0. \tag{2.10}
$$

By (2.7), this equation admits $(m-1)$ linearly independent solutions, and every corresponding function can be supported on two cells. Particularly, we assign the two cells to be adjacent. Figure 2 illustrates the profile of a basis function.

The function as illustrated in Figure 2, denoted by $\underset{\sim}{\tau}$, is

$$
\underset{\sim}{\tau} = \underset{\sim}{b}_{T_L}^M \text{ on } T_L,\ \ \underset{\sim}{\tau} = -\underset{\sim}{b}_{T_R}^M \text{ on } T_R, \text{ and } \underset{\sim}{\tau} = 0, \text{ elsewhere.}
$$

By (2.7), on T_L, $(\underset{\sim}{\tau}, \nabla\psi_M)_{T_L} + (\mathrm{div}\underset{\sim}{\tau}, \psi_M)_{T_L} = 1$, $(\underset{\sim}{\tau}, \nabla\psi_L)_{T_L} + (\mathrm{div}\underset{\sim}{\tau}, \psi_L)_{T_L} = 0$, and $(\underset{\sim}{\tau}, \nabla\psi_N)_{T_L} + (\mathrm{div}\underset{\sim}{\tau}, \psi_N)_{T_L} = 0$; on T_R, $(\underset{\sim}{\tau}, \nabla\psi_M)_{T_R} + (\mathrm{div}\underset{\sim}{\tau}, \psi_M)_{T_R} = -1$, $(\underset{\sim}{\tau}, \nabla\psi_R)_{T_R} + (\mathrm{div}\underset{\sim}{\tau}, \psi_R)_{T_R} = 0$, and $(\underset{\sim}{\tau}, \nabla\psi_N)_{T_R} + (\mathrm{div}\underset{\sim}{\tau}, \psi_N)_{T_R} = 0$. Then $\underset{\sim}{\tau}$ satisfies (2.10). As $\underset{\sim}{\tau}$ vanishes on other cells, we can obtain $\sum_{T\in\mathscr{T}_h} (\underset{\sim}{\tau}, \nabla\psi)_T + (\mathrm{div}\underset{\sim}{\tau}, \psi)_T = 0$ for all $\psi\in\mathbb{V}_h^1$, thus $\underset{\sim}{\tau}\in\mathbb{RT}_{h0}^{\rm nc}$.

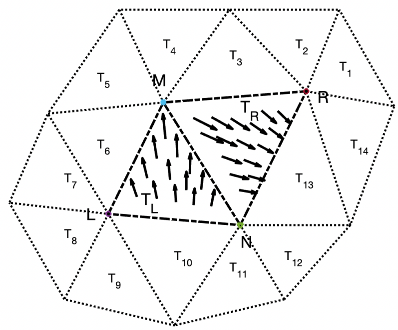

FIG. 2. Profile of the field of a **global** basis functions in $\mathscr{B}_M$, supported on two adjacent cells.

According to the profile of Figure 2, a set of linearly independent basis functions of $\mathscr{B}_M$ can be given in Figure 3, where M is an interior vertex, and in Figure 4, where M is a boundary vertex. This completes the proof. $\square$

Lemma 5 *Given a vertex A that is shared by m triangles, $\dim(\mathscr{C}_A) = m$. There exist a set of basis functions of $\mathscr{C}_A$, each of which is supported on just one cell.*

The proof of Lemma 5 is straightforward. We refer to Figure 5 for an illustration.

Remark 6 *For $\mathbb{RT}^{\rm nc}_{h0}$, the total amount of the locally supported basis functions is*

$$\sum_{M\in\mathscr{X}_h} [\#\{T\in\mathscr{T}_h : \partial T\ni M\}-1] = 3\#[T\in\mathscr{T}_h] - \#[M\in\mathscr{X}_h] = \dim(\mathbb{RT}^{\rm nc}_{h0}).$$

For $\mathbb{RT}^{\rm nc}_{h}$, the total amount of the locally supported basis functions is

$$\sum_{M\in\mathscr{X}_h^b} [\#\{T\in\mathscr{T}_h : \partial T\ni M\}] + \sum_{M\in\mathscr{X}_h^i} [\#\{T\in\mathscr{T}_h : \partial T\ni M\}-1]$$
$$= 3\#[T\in\mathscr{T}_h] - \#\left[M\in\mathscr{X}_h^i\right] = \dim(\mathbb{RT}^{\rm nc}_{h}).$$

In any case, T is covered by the supports of no more than $\widetilde{m}+6$ basis functions in $\mathbb{RT}^{\rm nc}_{h}$ or $\mathbb{RT}^{\rm nc}_{h0}$, where $\widetilde{m}$ is the number of cells that has at least one vertex in common with T. The generation of a local stiffness matrix is a local operation, and the assembling of global stiffness matrices can be done by following the standard routine for finite elements of Ciarlet-type.

Based on the specific profiles of the basis functions, we conclude this subsection by rephrasing Lemma 3 as the theorem below.

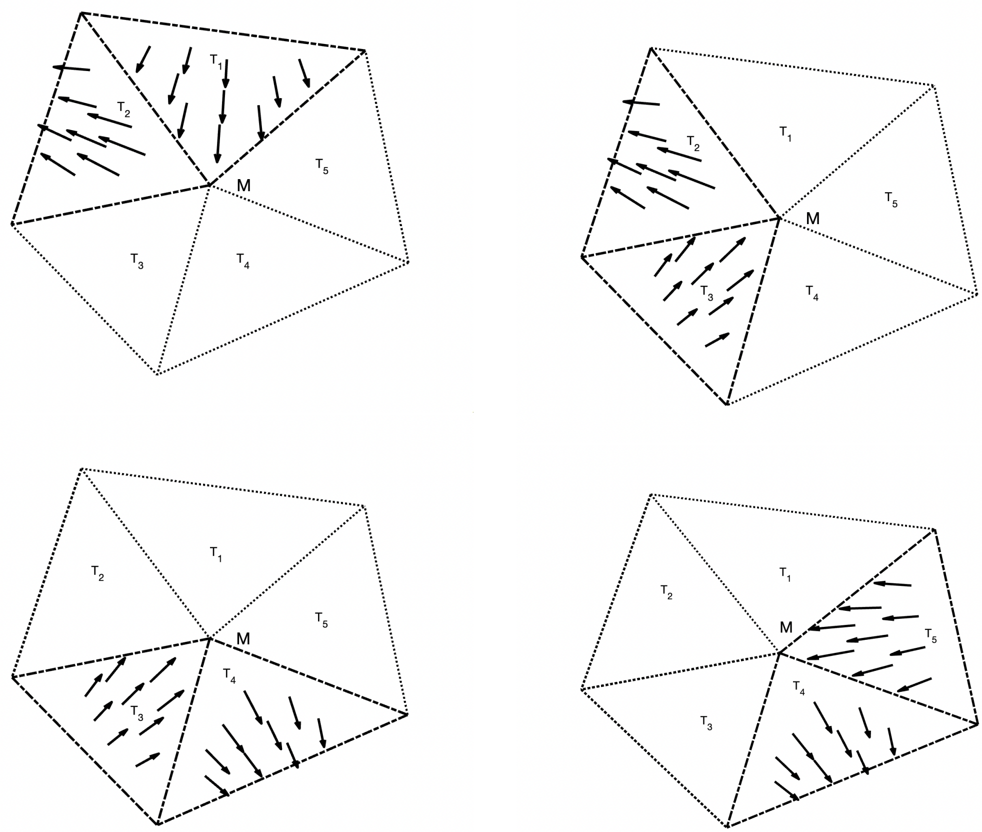

FIG. 3. Profiles of the fields of linearly independent basis functions of $\mathscr{B}_M$, $M \in \mathscr{X}_h^i$.

Theorem 7 *The space $\mathbb{RT}_{h0}^{\rm nc}$ admits a set of linear independent basis functions, which are belonging to $\bigoplus_{M\in\mathscr{X}_h}\mathscr{B}_M$ and each supported on two adjacent triangles.*

The space $\mathbb{RT}_{h}^{\rm nc}$ admits a set of linear independent basis functions; they consist of two types of functions, Type I and Type II. The functions of Type I are belonging to $\bigoplus_{M\in\mathscr{X}_h^i}\mathscr{B}_M$ and each supported on two adjacent triangles, and the functions of Type II are belonging to $\bigoplus_{M\in\mathscr{X}_h^b}\mathscr{C}_M$ and each supported on one triangle.

2.2. *Approximation and stability*

2.2.1. Locally-defined projective interpolator for $H(\mathrm{div})$
Given a triangle T, define the cell-wise interpolator

$$\mathbb{I}_T^{\rm RT} : H(\mathrm{div}, T) \to \mathbb{RT}(T) \tag{2.11}$$

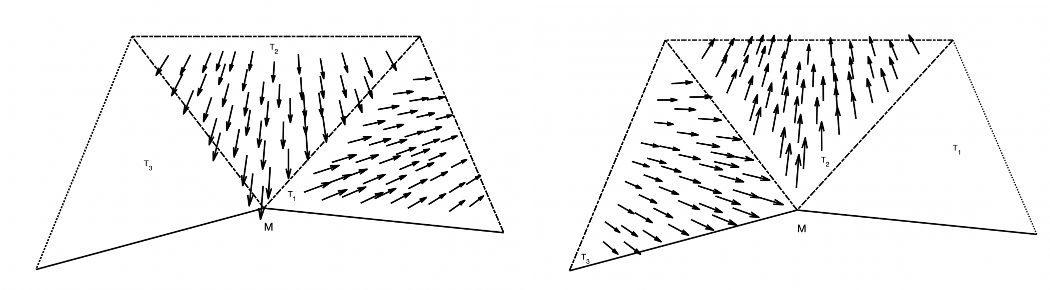

FIG. 4. Illustration of global basis functions of $\mathbb{RT}^{\rm nc}_{h0}$ based on a boundary vertex M

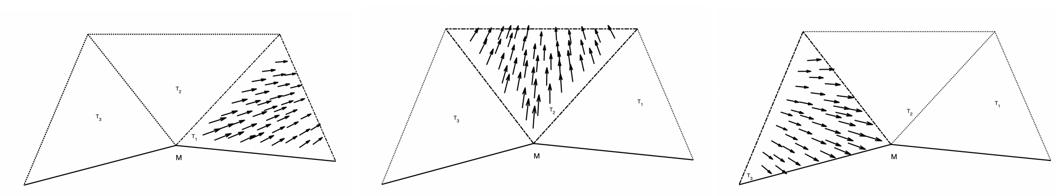

FIG. 5. The local basis functions associated with a boundary vertex M can work as global basis functions of $\mathbb{RT}^{\rm nc}_h$.

such that

$$(\mathbb{I}^{\rm RT}_T\underset{\sim}{\tau},\nabla v)_T+({\rm div}\mathbb{I}^{\rm RT}_T\underset{\sim}{\tau},v)_T=(\underset{\sim}{\tau},\nabla v)_T+({\rm div}\underset{\sim}{\tau},v)_T,\ \forall v\in P_1(T). \tag{2.12}$$

By (2.7), $\mathbb{I}^{\rm RT}_T\underset{\sim}{\tau}=\displaystyle\sum_{i=1}^{3}\left[(\underset{\sim}{\tau},\nabla\lambda_i)_T+({\rm div}\underset{\sim}{\tau},\lambda_i)_T\right]\underset{\sim}{b}{}^{a_i}_T$, and $\mathbb{I}^{\rm RT}_T\underset{\sim}{\sigma}=\underset{\sim}{\sigma}$ for $\underset{\sim}{\sigma}\in\mathbb{RT}(T)$.

Remark 8 *The Crouzeix-Raviart element interpolator $\mathbb{I}^{\rm CR}_T:H^1(T)\to P_1(T)$, defined such that $\int_e\mathbb{I}^{\rm CR}_Tv=\int_e v$, satisfies the condition $(\mathbb{I}^{\rm CR}_Tv,{\rm div}\underset{\sim}{\tau})_T+(\nabla\mathbb{I}^{\rm CR}_Tv,\underset{\sim}{\tau})_T=(v,{\rm div}\underset{\sim}{\tau})_T+(\nabla v,\underset{\sim}{\tau})_T,\ \forall\underset{\sim}{\tau}\in\mathbb{RT}(T)$.*

On the triangulation $\mathscr{T}_h$, define the global interpolator by

$$\mathbb{I}^{\rm RT}_h:\bigoplus_{T\in\mathscr{T}_h}E^{\Omega}_TH({\rm div},T)\to\mathbb{RT}(\mathscr{T}_h),\ \text{ such that }\ (\mathbb{I}^{\rm RT}_h\underset{\sim}{\tau}_h)_T=\mathbb{I}^{\rm RT}_T(\underset{\sim}{\tau}_h|_T),\ \forall T\in\mathscr{T}_h. \tag{2.13}$$

Note that $\mathbb{I}^{\rm RT}_h$ is defined completely cell by cell; namely, for any two functions $\underset{\sim}{v}_h$, $\underset{\sim}{w}_h\in\bigoplus_{T\in\mathscr{T}_h}E^{\Omega}_TH({\rm div},T)$ and any cell T such that $\underset{\sim}{v}_h=\underset{\sim}{w}_h$ on T, it holds that $\mathbb{I}^{\rm RT}_h\underset{\sim}{v}_h=\mathbb{I}^{\rm RT}_h\underset{\sim}{w}_h$ on T.

Lemma 9 *$\mathscr{R}(\mathbb{I}^{\rm RT}_h,H({\rm div},\Omega))\subset\mathbb{RT}^{\rm nc}_h$ and $\mathscr{R}(\mathbb{I}^{\rm RT}_h,H_0({\rm div},\Omega))\subset\mathbb{RT}^{\rm nc}_{h0}$.*

Proof Given $\underset{\sim}{\sigma}\in H({\rm div},\Omega)$, $(\underset{\sim}{\sigma},\nabla v_h)+({\rm div}\underset{\sim}{\sigma},v_h)=0$ for any $v_h\in\mathbb{V}^1_{h0}$. Thus for any $v_h\in\mathbb{V}^1_{h0}$, $\displaystyle\sum_{T\in\mathscr{T}_h}(\nabla v_h,\mathbb{I}^{\rm RT}_h\underset{\sim}{\sigma})_T+(v_h,{\rm div}\mathbb{I}^{\rm RT}_h\underset{\sim}{\sigma})_T=\sum_{T\in\mathscr{T}_h}(\nabla v_h,\underset{\sim}{\sigma})_T+(v_h,{\rm div}\underset{\sim}{\sigma})_T=0$. Namely $\mathbb{I}^{\rm RT}_h\underset{\sim}{\sigma}\in\mathbb{RT}^{\rm nc}_h$, and thus $\mathscr{R}(\mathbb{I}^{\rm RT}_h,H({\rm div},\Omega))\subset\mathbb{RT}^{\rm nc}_h$. Similarly $\mathscr{R}(\mathbb{I}^{\rm RT}_h,H_0({\rm div},\Omega))\subset\mathbb{RT}^{\rm nc}_{h0}$. This completes the proof. □

Remark 10 *Different from most existing interpolators, $\mathbb{I}_h^{\rm RT}\underset{\sim}{\sigma}$ is not defined in the form of $\sum l_i(\underset{\sim}{\sigma})\underset{\sim}{\tau}_i$, where $\underset{\sim}{\tau}_i$ is each a global basis function of $\mathbb{RT}_h^{\rm nc}$, and l_i is each a functional on $\underset{\sim}{\sigma}$. Indeed, according to theory of Zeng et al. (2023), as the global basis functions of $\mathbb{RT}_h^{\rm nc}$ may not be locally linearly independent, interpolator defined as $\sum l_i(\underset{\sim}{\sigma})\underset{\sim}{\tau}_i$ with l_i depends on the local information of $\underset{\sim}{\sigma}$ cannot be projective.*

2.2.2. Approximation and stability

Lemma 11 *With a constant C depending on the shape regularity of T,*

1. *stabilities:*
$$\|{\rm div}\mathbb{I}_T^{\rm RT}\underset{\sim}{\sigma}\|_{0,T}\leqslant\|{\rm div}\underset{\sim}{\sigma}\|_{0,T},\ \ and\ \ \ \|\mathbb{I}_T^{\rm RT}\underset{\sim}{\sigma}\|_{{\rm div},T}\leqslant C\|\underset{\sim}{\sigma}\|_{{\rm div},T};$$
2. *optimal approximation:*
$$\|{\rm div}(\underset{\sim}{\sigma}-\mathbb{I}_T^{\rm RT}\underset{\sim}{\sigma})\|_{0,T}=\inf_{\underset{\sim}{\tau}\in\mathbb{RT}(T)}\|{\rm div}(\underset{\sim}{\sigma}-\underset{\sim}{\tau})\|_{0,T},\ \ and\ \ \ \|\underset{\sim}{\sigma}-\mathbb{I}_T^{\rm RT}\underset{\sim}{\sigma}\|_{{\rm div},T}\leqslant C\inf_{\underset{\sim}{\tau}\in\mathbb{RT}(T)}\|\underset{\sim}{\sigma}-\underset{\sim}{\tau}\|_{{\rm div},T}.$$

Proof Evidently, ${\rm div}\mathbb{I}_T^{\rm RT}\underset{\sim}{\sigma}$ is the $L^2(T)$ projection of ${\rm div}\underset{\sim}{\sigma}$ onto piecewise constant space; therefore, $\|{\rm div}\mathbb{I}_T^{\rm RT}\underset{\sim}{\sigma}\|_{0,T}\leqslant\|{\rm div}\underset{\sim}{\sigma}\|_{0,\Omega}$. Now we use $\mathbf{P}_T^0$ for the $L^2(T)$ projection to constant, and $\underset{\sim}{\mathbf{P}}_T^0:=(\mathbf{P}_T^0)^2$. Then for any $\underset{\sim}{\tau}\in\mathbb{RT}(T)$, we have by Poincaré inequality, $\|\underset{\sim}{\tau}-\underset{\sim}{\mathbf{P}}_T^0\underset{\sim}{\tau}\|_{0,T}\leqslant Ch_T\|\nabla\underset{\sim}{\tau}\|_{0,T}=Ch_T/\sqrt{2}\|{\rm div}\underset{\sim}{\tau}\|_{0,T}$. Meanwhile, for any $v\in P_1(T)$, $\|v-\mathbf{P}_T^0v\|_{0,T}\leqslant Ch_T\|\nabla v\|_{0,T}$. For any $v\in P_1(T)$, $\left(\mathbb{I}_T^{\rm RT}\underset{\sim}{\sigma},\nabla v\right)_T+({\rm div}\mathbb{I}_T^{\rm RT}\underset{\sim}{\sigma},v)_T=(\underset{\sim}{\sigma},\nabla v)_T+({\rm div}\underset{\sim}{\sigma},v)_T$, and $({\rm div}\mathbb{I}_T^{\rm RT}\underset{\sim}{\sigma},\mathbf{P}_T^0v)_T=({\rm div}\underset{\sim}{\sigma},\mathbf{P}_T^0v)_T$. Therefore, $\left(\underset{\sim}{\mathbf{P}}_T^0\mathbb{I}_T^{\rm RT}\underset{\sim}{\sigma},\nabla v\right)_T=(\underset{\sim}{\sigma},\nabla(v-\mathbf{P}_T^0v))_T+({\rm div}\underset{\sim}{\sigma},v-\mathbf{P}_T^0v)_T$. It follows that $\|\underset{\sim}{\mathbf{P}}_T^0\mathbb{I}_T^{\rm RT}\underset{\sim}{\sigma}\|_{0,T}\leqslant C(\|\underset{\sim}{\sigma}\|_{0,T}+h_T\|{\rm div}\underset{\sim}{\sigma}\|_{0,T})$. Further $\|\mathbb{I}_T^{\rm RT}\underset{\sim}{\sigma}\|_{0,T}\leqslant C(\|\underset{\sim}{\sigma}\|_{0,T}+h_T\|{\rm div}\underset{\sim}{\sigma}\|_{0,T})$, and $\|\mathbb{I}_T^{\rm RT}\underset{\sim}{\sigma}\|_{{\rm div},T}\leqslant C\|\underset{\sim}{\sigma}\|_{{\rm div},T}$.

The optimal approximation follows then from the stability and by the standard procedure. □

Moreover, as the global interpolator is defined completely piecewise, global stabilities hold and
$$\|\underset{\sim}{\sigma}-\mathbb{I}_h^{\rm RT}\underset{\sim}{\sigma}\|_{{\rm div}_h}\leqslant C\inf_{\underset{\sim}{\tau}_h\in\mathbb{RT}(\mathscr{T}_h)}\|\underset{\sim}{\sigma}-\underset{\sim}{\tau}_h\|_{{\rm div}_h},\tag{2.14}$$
where C depends on the regularity of the triangulation only.

Further, the Poincaré inequalities hold for $\mathbb{RT}_h^{\rm nc}$ and $\mathbb{RT}_{h0}^{\rm nc}$.

Lemma 12 *Given $\underset{\sim}{\tau}_h\in\mathbb{RT}_h^{\rm nc}$, there is $\underset{\sim}{\sigma}_h\in\mathbb{RT}_h^{\rm nc}$, such that ${\rm div}_h\underset{\sim}{\sigma}_h={\rm div}_h\underset{\sim}{\tau}_h$, and $\|\underset{\sim}{\sigma}_h\|_{0,\Omega}\leqslant C\|{\rm div}_h\underset{\sim}{\tau}_h\|_{0,\Omega}$.*

Proof Note that ${\rm div}_h\underset{\sim}{\tau}_h$ is piecewise constant, and there exists a $\underset{\sim}{\tau}\in H({\rm div},\Omega)$, such that ${\rm div}\underset{\sim}{\tau}={\rm div}_h\underset{\sim}{\tau}_h$, and $\|\underset{\sim}{\tau}\|_{{\rm div},\Omega}\leqslant C\|{\rm div}\underset{\sim}{\tau}\|_{0,\Omega}$. Set $\underset{\sim}{\sigma}_h=\mathbb{I}_h^{\rm RT}\underset{\sim}{\tau}$, then ${\rm div}_h\underset{\sim}{\sigma}_h={\rm div}\underset{\sim}{\tau}$, and $\|\underset{\sim}{\sigma}_h\|_{{\rm div}_h}\leqslant C\|\underset{\sim}{\tau}\|_{\rm div}\leqslant C\|{\rm div}_h\underset{\sim}{\tau}_h\|_{0,\Omega}$. This completes the proof. □

Remark 13 *Evidently, $\mathbb{RT}_h^{\rm nc}\supset\underset{\sim}{V}_h^{\rm RT}$, and thus the approximation and stability properties of $\mathbb{RT}_h^{\rm nc}$ follow. Though, we present direct proofs of them by the aid of the interpolator. In some sense, both the two properties established here are optimal.*

2.3. *Discretization of the variational problems*

2.3.1. Discretization of the $H(\mathrm{div})$ elliptic problem
We consider the problem: given $\underset{\sim}{f} \in \underset{\sim}{L}^2(\Omega)$, find $\underset{\sim}{\sigma} \in H(\mathrm{div},\Omega)$, such that

$$(\mathrm{div}\underset{\sim}{\sigma},\mathrm{div}\underset{\sim}{\tau})+(\underset{\sim}{\sigma},\underset{\sim}{\tau})=(\underset{\sim}{f},\underset{\sim}{\tau}),\ \ \forall\,\underset{\sim}{\tau}\in H(\mathrm{div},\Omega). \tag{2.15}$$

It follows that $\mathrm{div}\underset{\sim}{\sigma} \in H^1_0(\Omega)$, and $\underset{\sim}{f} = -\nabla\mathrm{div}\underset{\sim}{\sigma}+\underset{\sim}{\sigma}$.

We here consider the discretization of (2.15): to find $\underset{\sim}{\sigma}_h \in \mathbb{RT}^{\mathrm{nc}}_h$, such that

$$(\mathrm{div}_h\underset{\sim}{\sigma}_h,\mathrm{div}_h\underset{\sim}{\tau}_h)+(\underset{\sim}{\sigma}_h,\underset{\sim}{\tau}_h)=(\underset{\sim}{f},\underset{\sim}{\tau}_h),\ \ \forall\,\underset{\sim}{\tau}_h\in \mathbb{RT}^{\mathrm{nc}}_h. \tag{2.16}$$

Immediately (2.15) and (2.16) are well-posed. Denote $\|\underset{\sim}{\tau}_h\|_{\mathrm{div}_h} := (\|\underset{\sim}{\tau}_h\|_0^2+\|\mathrm{div}_h\underset{\sim}{\tau}_h\|_0^2)^{1/2}$.

Theorem 14 *Let $\underset{\sim}{\sigma}$ and $\underset{\sim}{\sigma}_h$ be the solutions of* (2.15) *and* (2.16)*, respectively. Then*

$$\|\underset{\sim}{\sigma}-\underset{\sim}{\sigma}_h\|_{\mathrm{div}_h}\leqslant 2\inf_{\underset{\sim}{\tau}_h\in\mathbb{RT}^{\mathrm{nc}}_h}\|\underset{\sim}{\sigma}-\underset{\sim}{\tau}_h\|_{\mathrm{div}_h}+\inf_{v_h\in\mathbb{V}^1_{h0}}\|\mathrm{div}\underset{\sim}{\sigma}-v_h\|_{1,\Omega}. \tag{2.17}$$

Proof By Strang's lemma (cf. Ciarlet (1978)),

$$\|\underset{\sim}{\sigma}-\underset{\sim}{\sigma}_h\|_{\mathrm{div}_h}\leqslant 2\inf_{\underset{\sim}{\tau}_h\in\mathbb{RT}^{\mathrm{nc}}_h}\|\underset{\sim}{\sigma}-\underset{\sim}{\tau}_h\|_{\mathrm{div}_h}+\sup_{\underset{\sim}{\tau}_h\in\mathbb{RT}^{\mathrm{nc}}_h}\frac{(\mathrm{div}\underset{\sim}{\sigma},\mathrm{div}_h\underset{\sim}{\tau}_h)+(\nabla\mathrm{div}\underset{\sim}{\sigma},\underset{\sim}{\tau}_h)}{\|\underset{\sim}{\tau}_h\|_{\mathrm{div}_h}}.$$

For any $v_h\in\mathbb{V}^1_{h0}$,

$$(\mathrm{div}\underset{\sim}{\sigma},\mathrm{div}_h\underset{\sim}{\tau}_h)+(\nabla\mathrm{div}\underset{\sim}{\sigma},\underset{\sim}{\tau}_h)=(\mathrm{div}\underset{\sim}{\sigma}-v_h,\mathrm{div}_h\underset{\sim}{\tau}_h)+(\nabla(\mathrm{div}\underset{\sim}{\sigma}-v_h),\underset{\sim}{\tau}_h)\leqslant\|\mathrm{div}\underset{\sim}{\sigma}-v_h\|_{1,\Omega}\|\underset{\sim}{\tau}_h\|_{\mathrm{div}_h}.$$

Then (2.17) follows. □

By the abstract estimation, the precise convergence order can be figured out with respect to the assumption on the regularity of the solution.

2.3.2. Discretization of the Darcy problem
We consider the problem: given $f\in L^2(\Omega)$, find $(u,\underset{\sim}{\sigma})\in L^2(\Omega)\times H(\mathrm{div},\Omega)$, such that

$$\left\{\begin{array}{llll}(\underset{\sim}{\sigma},\underset{\sim}{\tau}) & +(u,\mathrm{div}\underset{\sim}{\tau}) & =0 & \forall\,\underset{\sim}{\tau}\in H(\mathrm{div},\Omega),\\ (\mathrm{div}\underset{\sim}{\sigma},v) & & =(f,v) & \forall\,v\in L^2(\Omega).\end{array}\right. \tag{2.18}$$

The discretization is to find $(u_h,\underset{\sim}{\sigma}_h)\in\mathscr{P}_0(\mathscr{T}_h)\times\mathbb{RT}^{\mathrm{nc}}_h$, such that

$$\left\{\begin{array}{llll}(\underset{\sim}{\sigma}_h,\underset{\sim}{\tau}_h) & +(u_h,\mathrm{div}_h\underset{\sim}{\tau}_h) & =0 & \forall\,\underset{\sim}{\tau}_h\in\mathbb{RT}^{\mathrm{nc}}_h,\\ (\mathrm{div}_h\underset{\sim}{\sigma}_h,v_h) & & =(f,v_h) & \forall\,v_h\in\mathscr{P}_0(\mathscr{T}_h).\end{array}\right. \tag{2.19}$$

Here $\mathscr{P}_0(\mathscr{T}_h)$ is the space of piecewise constant functions. Evidently, (2.19) is well-posed.

Theorem 15 *Let* $(u,\underset{\sim}{\sigma})$ *and* $(u_h,\underset{\sim}{\sigma}_h)$ *be the solutions of* (2.18) *and* (2.19)*, respectively. Then*

$$\|u-u_h\|_{0,\Omega}+\|\underset{\sim}{\sigma}-\underset{\sim}{\sigma}_h\|_{\mathrm{div}_h}\leqslant C\left[\inf_{\substack{v_h\in\mathscr{P}_0(\mathscr{T}_h)\\ \underset{\sim}{\tau}_h\in\mathbb{RT}_h^{\mathrm{nc}}}}(\|u-v_h\|_{0,\Omega}+\|\underset{\sim}{\sigma}-\underset{\sim}{\tau}_h\|_{\mathrm{div}_h})+\inf_{s_h\in\mathbb{V}_{h0}^1}\|u-s_h\|_{1,\Omega}\right].$$

Proof By the Strang lemma for saddle point problem (cf., e.g., Boffi et al. (2013, Proposition5.5.6)),

$$\|u-u_h\|_{0,\Omega}+\|\underset{\sim}{\sigma}-\underset{\sim}{\sigma}_h\|_{\mathrm{div}_h}\leqslant C\left[\inf_{\substack{v_h\in\mathscr{P}_0(\mathscr{T}_h)\\ \underset{\sim}{\tau}_h\in\mathbb{RT}_h^{\mathrm{nc}}}}(\|u-v_h\|_{0,\Omega}+\|\underset{\sim}{\sigma}-\underset{\sim}{\tau}_h\|_{\mathrm{div}_h})+\sup_{\underset{\sim}{\tau}_h\in\mathbb{RT}_h^{\mathrm{nc}}}\frac{(\underset{\sim}{\sigma},\underset{\sim}{\tau}_h)+(u,\mathrm{div}_h\underset{\sim}{\tau}_h)}{\|\underset{\sim}{\tau}_h\|_{\mathrm{div}_h}}\right].$$

Note that $\underset{\sim}{\sigma}=\nabla u$, and we have, for any $s_h\in\mathbb{V}_{h0}^1$,

$$(\underset{\sim}{\sigma},\underset{\sim}{\tau}_h)+(u,\mathrm{div}_h\underset{\sim}{\tau}_h)=(\nabla u-\nabla s_h,\underset{\sim}{\tau}_h)+(u-s_h,\mathrm{div}_h\underset{\sim}{\tau}_h)\leqslant\|u-s_h\|_{1,\Omega}\|\underset{\sim}{\tau}_h\|_{\mathrm{div}_h}.$$

It follows then

$$\|u-u_h\|_{0,\Omega}+\|\underset{\sim}{\sigma}-\underset{\sim}{\sigma}_h\|_{\mathrm{div}_h}\leqslant C\left[\inf_{\substack{v_h\in\mathscr{P}_0(\mathscr{T}_h)\\ \underset{\sim}{\tau}_h\in\mathbb{RT}_h^{\mathrm{nc}}}}(\|u-v_h\|_{0,\Omega}+\|\underset{\sim}{\sigma}-\underset{\sim}{\tau}_h\|_{\mathrm{div}_h})+\inf_{s_h\in\mathbb{V}_{h0}^1}\|u-s_h\|_{1,\Omega}\right].$$

This completes the proof. □

2.4. *Numerical experiments*

We show the implementability of $\mathbb{RT}_h^{\mathrm{nc}}$ and its difference from the classical Raviart-Thomas element by two series of experiments.

2.4.1. Implementability of the space $\mathbb{RT}_h^{\mathrm{nc}}$

Firstly, we use $\mathbb{RT}_h^{\mathrm{nc}}$ to solve numerically the boundary value problems (2.15) and (2.18). We use the unit square $(0,1)^2$ as the computation domain, and we choose properly the source terms, such that

- for (2.15), the exact solution is
$$\underset{\sim}{\sigma}=(-2\cos(\pi x)\sin(\pi y),\sin(\pi x)\cos(\pi y))^\top;$$
- for (2.18), the exact solution is
$$u=\sin(\pi x)\sin(\pi y),\ \text{ and }\ \underset{\sim}{\sigma}=\nabla u.$$

We construct two series of triangulations, being crisscross (cf. Figure 6, left) and irregular (cf. Figure 6, right), respectively. The computational results are recorded in Figures 7 and 8.

On the two series of triangulations, we also use $\mathbb{RT}_h^{\mathrm{nc}}\times\mathscr{P}_0(\mathscr{T}_h)$ to solve the eigenvalue problem of (2.18), which is to find $(\lambda,u,\underset{\sim}{\sigma})\in\mathbb{R}\times L^2(\Omega)\times H(\mathrm{div},\Omega)$, such that

$$\begin{cases}(\underset{\sim}{\sigma},\underset{\sim}{\tau}) & +(u,\mathrm{div}\underset{\sim}{\tau}) & =0 & \forall\underset{\sim}{\tau}\in H(\mathrm{div},\Omega),\\ (\mathrm{div}\underset{\sim}{\sigma},v) & & =\lambda\ \pi^2\ (u,v) & \forall v\in L^2(\Omega).\end{cases}\tag{2.20}$$

Note that on unit square, the eigenvalues of (2.20) take the values m^2+n^2, $m,n\in\mathbb{N}^+$. Here we separate the effect of π^2 so that the results are easy to read. The respective eigenvalue problems of (2.15) and

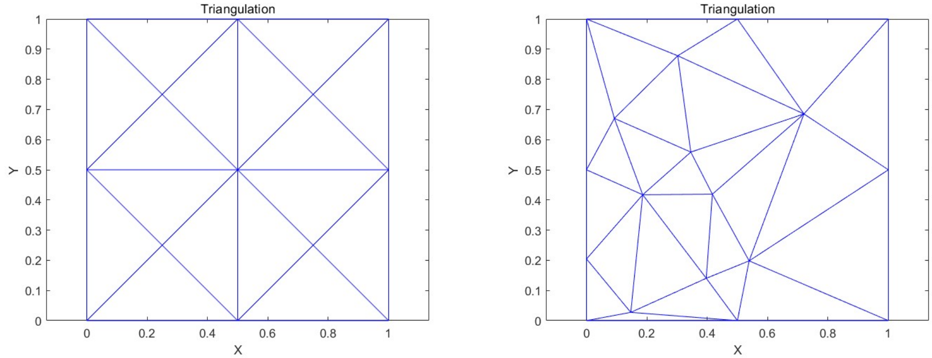

FIG. 6. The initial triangulation of two series of triangulations. Left: crisscross; right: irregular.

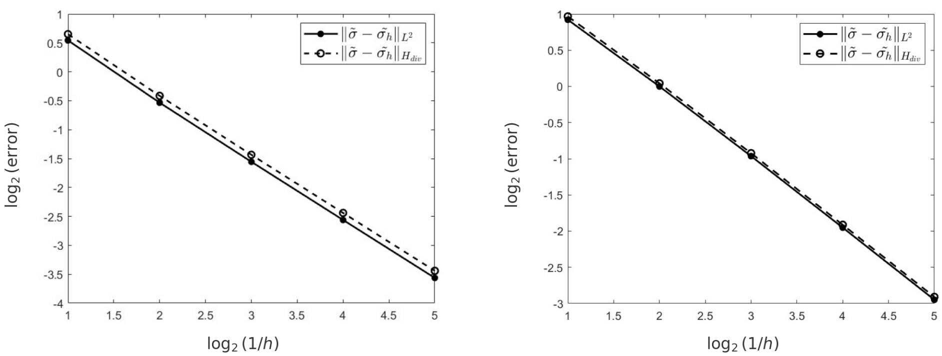

FIG. 7. Convergence process for (2.15). Left: on crisscross triangulations; right: on irregular triangulations.

(2.18) are essentially equivalent to each other. The 10 smallest computed eigenvalues of (2.20) on each series of grids are recorded in Tables 1 and 2. In the tables, we use "L" to denote the level of each grid, and use $\searrow$/$\nearrow$ to denote the decreasing/increasing trend of the computed eigenvalues as the grids are refined and refined. The computed eigenvalues converge to the exact eigenvalues nicely. Moreover, it can be seen that the $\mathbb{RT}_h^{\rm nc}$ scheme for (2.20) provides upper bounds to the exact eigenvalues.

L	λ_h^1	λ_h^2	λ_h^3	λ_h^4	λ_h^5	λ_h^6	λ_h^7	λ_h^8	λ_h^9	λ_h^{10}
1	2.619	9.727	9.727	9.727	19.123	29.181	29.181	29.181	29.181	29.181
2	2.128	5.982	5.982	10.477	14.547	14.547	20.650	20.650	32.039	38.907
3	2.031	5.223	5.223	8.511	11.009	11.009	14.480	14.480	20.137	20.137
4	2.008	5.055	5.055	8.122	10.242	10.242	13.345	13.345	17.739	17.739
5	2.002	5.014	5.014	8.030	10.060	10.060	13.085	13.085	17.182	17.182
	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$

TABLE 1 *Computed eigenvalues by $\mathbb{RT}_h^{\rm nc}$ scheme on crisscross grids.*

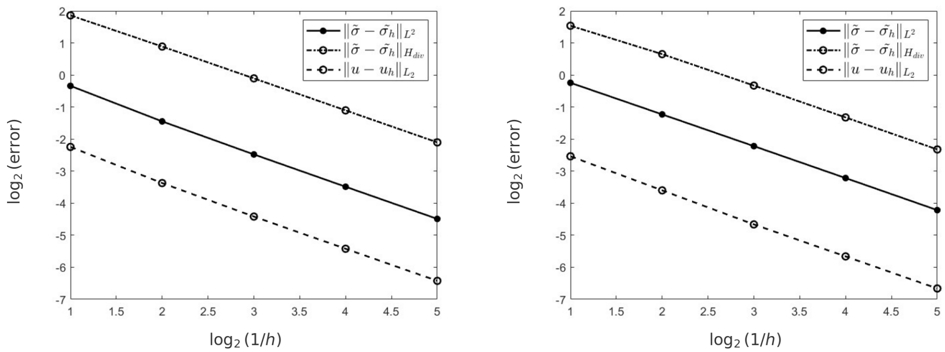

FIG. 8. Convergence process for (2.18). Left: on crisscross triangulations; right: on irregular triangulations.

L	λ_h^1	λ_h^2	λ_h^3	λ_h^4	λ_h^5	λ_h^6	λ_h^7	λ_h^8	λ_h^9	λ_h^{10}
1	2.474	6.921	7.575	12.150	17.167	20.663	21.687	23.653	25.825	26.359
2	2.123	5.636	5.770	9.850	12.660	13.141	18.222	18.582	24.122	25.578
3	2.031	5.164	5.199	8.474	10.712	10.778	14.307	14.357	18.985	19.218
4	2.008	5.041	5.050	8.119	10.182	10.195	13.325	13.336	17.520	17.565
5	2.002	5.010	5.013	8.030	10.046	10.049	13.081	13.084	17.132	17.142
	↘	↘	↘	↘	↘	↘	↘	↘	↘	↘

TABLE 2 *Computed eigenvalues by $\mathbb{RT}_h^{\mathrm{nc}}$ scheme on irregular grids.*

2.4.2. Comparison with the classical Raviart-Thomas element

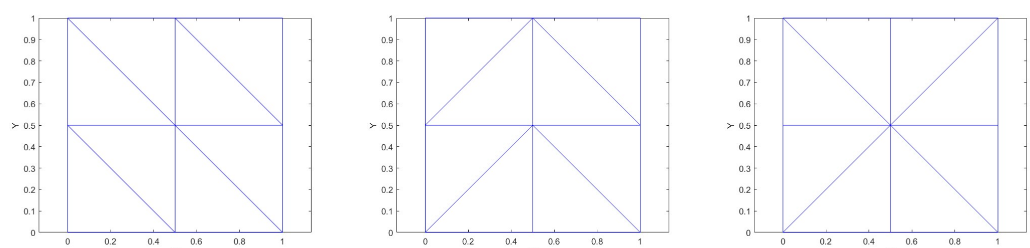

FIG. 9. The initial triangulations. Left: regular; middle: fish bone; right: union Jack.

We here show the experiments of solving the eigenvalue problem (2.20) with the classical Raviart-Thomas element scheme on the crisscross triangulation, the regular triangulation (cf. Figure 9, left), the fish-bone triangulation (cf. Figure 9, middle), and the union Jack triangulation (cf. Figure 9, right). The 10 smallest computed eigenvalues on each series of grids are recorded in Tables 3, 4, 5 and 6. It can be seen that the classical (lowest-degree) Raviart-Thomas element might provide upper or lower bound to different eigenvalues, sensitive to the grid as well.[2] Numerical experiments on these special triangulations which are easy to check are included.

L	λ_h^1	λ_h^2	λ_h^3	λ_h^4	λ_h^5	λ_h^6	λ_h^7	λ_h^8	λ_h^9	λ_h^{10}
1	1.858	4.158	4.158	8.254	9.727	12.042	12.042	12.733	14.590	14.590
2	1.965	4.893	4.893	7.431	9.850	9.850	11.731	11.731	14.847	15.317
3	1.991	4.975	4.975	7.862	9.986	9.986	12.712	12.712	17.071	17.071
4	1.998	4.994	4.994	7.966	9.998	9.998	12.929	12.929	17.024	17.024
5	1.999	4.998	4.998	7.991	9.999	9.999	12.982	12.982	17.006	17.006
	↗	↗	↗	↗	↗	↗	↗	↗	↘	↘

TABLE 3 *Computed eigenvalues by the classical Raviart-Thomas element scheme on crisscross grids.*

L	λ_h^1	λ_h^2	λ_h^3	λ_h^4	λ_h^5	λ_h^6	λ_h^7	λ_h^8	λ_h^9	λ_h^{10}
1	2.110	3.542	4.863	9.727	9.727	12.021	13.453	14.590	—	—
2	2.032	4.834	5.096	8.077	8.957	9.414	11.107	11.377	12.242	14.729
3	2.008	4.964	5.026	8.119	9.798	9.815	12.896	13.422	16.153	16.196
4	2.002	4.991	5.007	8.033	9.951	9.952	12.983	13.113	16.791	16.799
5	2.001	4.998	5.002	8.009	9.988	9.988	12.996	13.029	16.947	16.950
	↘	↗	↘	↘	↗	↗	↗	↘	↗	↗

TABLE 4 *Computed eigenvalues by the classical Raviart-Thomas element scheme on regular grids.*

L	λ_h^1	λ_h^2	λ_h^3	λ_h^4	λ_h^5	λ_h^6	λ_h^7	λ_h^8	λ_h^9	λ_h^{10}
1	2.084	4.127	4.127	9.727	9.727	12.895	12.895	14.590	—	—
2	2.032	4.943	4.959	8.337	8.881	8.989	11.359	11.501	12.716	13.188
3	2.008	4.993	4.995	8.126	9.788	9.800	13.153	13.166	16.107	16.159
4	2.002	4.999	4.999	8.034	9.950	9.951	13.047	13.048	16.790	16.794
5	2.001	5.000	5.000	8.009	9.988	9.988	13.012	13.012	16.948	16.948
	↘	↗	↗	↘	↗	↗	↘	↘	↗	↗

TABLE 5 *Computed eigenvalues by the classical Raviart-Thomas element scheme on fish-bone grids.*

The $\mathbb{RT}_h^{\rm nc}$ scheme for (2.20) is further carried out on the regular triangulation, fish-bone triangulation and the union Jack triangulation, and the 10 smallest computed eigenvalues on each series of grids are recorded in Tables 7, 8 and 9. It can be seen that, in all these experiments, again, the $\mathbb{RT}_h^{\rm nc}$ scheme for (2.20) provides upper bounds for all the eigenvalues. The robustness is improved with $\mathbb{RT}_h^{\rm nc}$. This will be further investigated in future.

[2] We do not think it is now found for the first time that the classical (lowest-degree) Raviart-Thomas element scheme cannot be expected to provide a certain bounds to the exact eigenvalues, though we do not find a referred literature.

L	λ_h^1	λ_h^2	λ_h^3	λ_h^4	λ_h^5	λ_h^6	λ_h^7	λ_h^8	λ_h^9	λ_h^{10}
1	2.432	4.127	4.127	7.295	9.727	12.895	12.895	14.590	—	—
2	2.030	4.925	4.925	8.315	9.727	9.727	11.501	11.501	13.497	13.497
3	2.008	4.993	4.993	8.120	9.786	9.786	13.133	13.133	16.097	16.097
4	2.002	4.999	4.999	8.033	9.950	9.950	13.047	13.047	16.789	16.789
5	2.001	5.000	5.000	8.009	9.988	9.988	13.012	13.012	16.948	16.948
	$\searrow$	$\nearrow$	$\nearrow$	$\searrow$	$\nearrow$	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$	$\nearrow$

TABLE 6 *Computed eigenvalues by the classical Raviart-Thomas element scheme on union Jack grids.*

L	λ_h^1	λ_h^2	λ_h^3	λ_h^4	λ_h^5	λ_h^6	λ_h^7	λ_h^8	λ_h^9	λ_h^{10}
1	3.648	14.590	14.590	14.590	14.590	14.590	14.590	14.590	—	—
2	2.396	6.748	8.210	13.339	19.454	21.970	23.399	33.381	36.189	58.361
3	2.095	5.414	5.692	9.432	12.082	12.343	15.678	18.242	23.299	23.656
4	2.024	5.102	5.166	8.372	10.494	10.510	13.684	14.246	18.387	18.430
5	2.006	5.026	5.041	8.094	10.122	10.123	13.173	13.306	17.335	17.344
	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$

TABLE 7 *Computed eigenvalues by $\mathbb{RT}_h^{\mathrm{nc}}$ scheme on regular grids.*

L	λ_h^1	λ_h^2	λ_h^3	λ_h^4	λ_h^5	λ_h^6	λ_h^7	λ_h^8	λ_h^9	λ_h^{10}
1	3.648	14.590	14.590	14.590	14.590	14.590	14.590	14.590	—	—
2	2.395	7.247	7.455	14.590	17.639	20.437	26.875	32.313	36.332	58.361
3	2.095	5.537	5.552	9.559	11.969	12.131	16.941	17.131	22.453	23.322
4	2.024	5.133	5.134	8.380	10.485	10.497	13.960	13.973	18.334	18.398
5	2.006	5.033	5.033	8.094	10.121	10.122	13.239	13.240	17.334	17.339
	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$

TABLE 8 *Computed eigenvalues by $\mathbb{RT}_h^{\mathrm{nc}}$ scheme on fish-bone grids.*

L	λ_h^1	λ_h^2	λ_h^3	λ_h^4	λ_h^5	λ_h^6	λ_h^7	λ_h^8	λ_h^9	λ_h^{10}
1	2.918	14.590	14.590	14.590	14.590	14.590	14.590	14.590	—	—
2	2.366	7.274	7.274	11.672	19.454	19.454	29.531	29.531	43.615	58.361
3	2.087	5.505	5.505	9.466	11.963	11.963	16.852	16.852	22.973	22.973
4	2.022	5.121	5.121	8.349	10.447	10.447	13.893	13.893	18.258	18.258
5	2.005	5.030	5.030	8.086	10.109	10.109	13.218	13.218	17.301	17.301
	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$	$\searrow$

TABLE 9 *Computed eigenvalues by $\mathbb{RT}_h^{\mathrm{nc}}$ scheme on union Jack grids.*

3. A nonconforming framework for finite element exterior calculus

3.1. *Nonconforming finite element spaces for $H\Lambda^k$ in $\mathbb{R}^n$*

Let $\mathscr{G}_h$ be a simplicial subdivision of Ω. For $0 \leqslant k \leqslant n-1$, we define finite element spaces for $H\Lambda^k$ by

$$\mathbf{W}_h^{\mathrm{nc}}\Lambda^k := \left\{\omega_h \in \mathscr{P}_1^-\Lambda^k(\mathscr{G}_h) : \langle \omega_h, \delta_{k+1}\eta_h\rangle_{L^2\Lambda^k} - \langle \mathbf{d}_h^k\omega_h, \eta_h\rangle_{L^2\Lambda^{k+1}} = 0,\ \forall\, \eta_h \in \mathbf{W}_{h0}^*\Lambda^{k+1}\right\}, \quad (3.1)$$

where $\mathbf{d}_h^k$ denotes the cell-by-cell operation of $\mathbf{d}^k$, and, for $H_0\Lambda^k$,

$$\mathbf{W}_{h0}^{\rm nc}\Lambda^k := \left\{\omega_h \in \mathscr{P}_1^-\Lambda^k(\mathscr{G}_h) : \langle \omega_h, \delta_{k+1}\eta_h\rangle_{L^2\Lambda^k} - \langle \mathbf{d}_h^k\omega_h, \eta_h\rangle_{L^2\Lambda^{k+1}} = 0,\ \forall \eta_h \in \mathbf{W}_h^*\Lambda^{k+1}\right\}. \quad (3.2)$$

Set

$$\mathbf{W}_h^{\rm nc}\Lambda^n := \mathscr{P}_0\Lambda^n(\mathscr{G}_h), \quad \text{and} \quad \mathbf{W}_{h0}^{\rm nc}\Lambda^n := \mathbf{W}_h^{\rm nc}\Lambda^n \cap L_0^2\Lambda^n(\Omega). \quad (3.3)$$

Remark 16 *Note that $\mathbf{W}_h^{\rm nc}\Lambda^0$ and $\mathbf{W}_{h0}^{\rm nc}\Lambda^0$ are the lowest-degree Crouzeix-Raviart element spaces. If further $n=1$, $\mathbf{W}_h^{\rm nc}\Lambda^0$ and $\mathbf{W}_{h0}^{\rm nc}\Lambda^0$ coincide with the respective continuous linear element spaces.*

Remark 17 *Associated with the definitions, by (1.3), it holds that, for example,*

$$\mathbf{W}_h^*\Lambda^k = \left\{\mu_h \in \mathscr{P}_1^{*,-}\Lambda^k(\mathscr{G}_h) : \langle \delta_k\mu_h, \tau_h\rangle_{L^2\Lambda^{k-1}} - \langle \mu_h, \mathbf{d}_h^{k-1}\tau_h\rangle_{L^2\Lambda^k} = 0,\ \forall \tau_h \in \mathbf{W}_{h0}^{\rm nc}\Lambda^{k-1}\right\}.$$

By the same virtue of Theorem 7, noting (1.3), we can prove theorem below.

Theorem 18 *The space $\mathbf{W}_{h0}^{\rm nc}\Lambda^k$ admits a set of linear independent basis functions, which are each supported on two adjacent simplices.*

The space $\mathbf{W}_h^{\rm nc}\Lambda^k$ admits a set of linear independent basis functions; they consist of two types of functions, Type I and Type II. The functions of Type I are each supported on two adjacent simplices, and the functions of Type II are each supported on one simplex.

In the sequel, we use $\mathscr{F}^{\mathscr{G}}$ for a family of shape regular subdivisions of Ω.

3.1.1. Locally defined interpolator and optimal approximation

Similar to (2.11), we define a local interpolator $\mathbb{I}_T^{\mathbf{d}^k} : H\Lambda^k(T) \to \mathscr{P}_1^-\Lambda^k(T)$, $0 \leqslant k \leqslant n-1$, such that,

$$\langle \mathbb{I}_T^{\mathbf{d}^k}\omega, \delta_{k+1}\eta\rangle_{L^2\Lambda^k(T)} - \langle \mathbf{d}^k\mathbb{I}_T^{\mathbf{d}^k}\omega, \eta\rangle_{L^2\Lambda^{k+1}(T)} = \langle \omega, \delta_{k+1}\eta\rangle_{L^2\Lambda^k(T)} - \langle \mathbf{d}^k\omega, \eta\rangle_{L^2\Lambda^{k+1}(T)},$$

for any $\eta \in \mathscr{P}_1^{*,-}\Lambda^{k+1}(T)$, and, following (2.13), define a global interpolator

$$\mathbb{I}_h^{\mathbf{d}^k} : \bigoplus_{T\in\mathscr{G}_h} E_T^\Omega H\Lambda^k(T) \to \mathscr{P}_1^-\Lambda^k(\mathscr{G}_h), \text{ by } (\mathbb{I}_h^{\mathbf{d}^k}\omega)|_T = \mathbb{I}_T^{\mathbf{d}^k}(\omega|_T),\ \forall T \in \mathscr{G}_h.$$

Set $\mathbb{I}_T^{\mathbf{d}^n}$ the $L^2(T)$ projection to $\mathscr{P}_0\Lambda^n$ on T, and $\mathbb{I}_h^{\mathbf{d}^n}$ the $L^2(\Omega)$ projection to $\mathscr{P}_0\Lambda^n(\mathscr{G}_h)$. Again all the interpolators are local ones.

Denote $\|\mu_h\|_{\mathbf{d}_h^k} := (\|\mathbf{d}_h^k\mu_h\|_{L^2\Lambda^{k+1}}^2 + \|\mu_h\|_{L^2\Lambda^k}^2)^{1/2}$. The proofs of the two lemmas below are the same as that of Lemma 9 and Lemma 11, and are omitted here.

Lemma 19 *$\mathscr{R}(\mathbb{I}_h^{\mathbf{d}^k}, H\Lambda^k) \subset \mathbf{W}_h^{\rm nc}\Lambda^k$ and $\mathscr{R}(\mathbb{I}_h^{\mathbf{d}^k}, H_0\Lambda^k) \subset \mathbf{W}_{h0}^{\rm nc}\Lambda^k$.*

Lemma 20 *With $C_{k,n}$ uniform for $\mathscr{F}^{\mathscr{G}}$, for $\mathscr{G}_h \in \mathscr{F}^{\mathscr{G}}$ and $\omega \in \bigoplus_{T\in\mathscr{G}_h} E_T^\Omega H\Lambda^k(T)$,*

$$\|\omega - \mathbb{I}_h^{\mathbf{d}^k}\omega\|_{\mathbf{d}_h^k} \leqslant C_{k,n} \inf_{\eta_h\in\mathscr{P}_1^-\Lambda^k(\mathscr{G}_h)} \|\omega - \eta_h\|_{\mathbf{d}_h^k}.$$

3.1.2. Uniform discrete Poincaré inequalities

As generally $\mathscr{R}(\mathbf{d}_h^k,\mathbf{W}_h^{\rm nc}\Lambda^k)\not\subset H\Lambda^{k+1}(\Omega)$, we cannot simply repeat the proof of Lemma 12. We adopt an indirect approach by Lemma 25, which can be viewed a finite-dimensional analogue of the closed range theorem by a comparison to Theorem 44 in Section B.

Let $\mathbf{X}$ and $\mathbb{Y}$ be two Hilbert spaces. For $(\mathbf{T},\mathbf{D}):\mathbf{X}\to\mathbb{Y}$ a closed operator, denote

$$\mathbf{D}^{\lrcorner\mathbf{T}}:=\{\mathbf{v}\in\mathbf{D}:\langle\mathbf{v},\mathbf{w}\rangle_{\mathbf{X}}=0,\ \forall\,\mathbf{w}\in\mathscr{N}(\mathbf{T},\mathbf{D})\}.$$

Define the **Poincaré inequality's criterion** of $(\mathbf{T},\mathbf{D})$ as

$$\mathsf{pic}(\mathbf{T},\mathbf{D}):=\begin{cases}\displaystyle\sup_{0\neq\mathbf{v}\in\mathbf{D}^{\lrcorner\mathbf{T}}}\frac{\|\mathbf{v}\|_{\mathbf{X}}}{\|\mathbf{T}\mathbf{v}\|_{\mathbb{Y}}}, & \text{if } \mathbf{D}^{\lrcorner\mathbf{T}}\neq\{0\};\\ 0, & \text{if } \mathbf{D}^{\lrcorner\mathbf{T}}=\{0\}.\end{cases}\tag{3.4}$$

If $\mathsf{pic}(\mathbf{T},\mathbf{D})$ is finite, then the Poincaré inequality holds for $(\mathbf{T},\mathbf{D})$. It is further indeed the best constant of the Poincaré inequality. The index can be used for a criterion for closed range. We refer to, e.g., Arnold (2018, Lemma 3.6) for a proof of Lemma 21 up to little technical modification.

Lemma 21 *For $(\mathbf{T},\mathbf{D}):\mathbf{X}\to\mathbb{Y}$ a closed operator, $\mathscr{R}(\mathbf{T},\mathbf{D})$ is closed if and only if $\mathsf{pic}(\mathbf{T},\mathbf{D})<+\infty$.*

The main estimation is the theorem below, which presents the uniform Poincaré inequality on the orthogonal complement of $\mathscr{N}(\mathbf{d}_h^k,\mathbf{W}_h^{\rm nc}\Lambda^k)$ in $\mathbf{W}_h^{\rm nc}\Lambda^k$.

Theorem 22 *With a constant $C_{k,n}$ uniform for $\mathscr{F}^{\mathscr{G}}$,*

$$\mathsf{pic}(\mathbf{d}_h^k,\mathbf{W}_h^{\rm nc}\Lambda^k)\leqslant C_{k,n}.$$

We firstly present three lemmas below, and postpone their technical proofs to appendix Section A.

Lemma 23 $\mathsf{pic}(\mathbf{d}_h^k,\mathbf{W}_h^{\rm nc}\Lambda^k)\leqslant\mathsf{pic}(\delta_{k+1},\mathbf{W}_{h0}^*\Lambda^{k+1})+2\mathsf{pic}(\mathbf{d}_h^k,\mathscr{P}_1^-\Lambda^k(\mathscr{G}_h))$.

Lemma 24 $\mathsf{pic}(\mathbf{d}_h^k,\mathscr{P}_1^-\Lambda^k(\mathscr{G}_h))=\mathscr{O}(h)$.

Lemma 25 $\left|\mathsf{pic}(\delta_{k+1},\mathbf{W}_{h0}^*\Lambda^{k+1})-\mathsf{pic}(\mathbf{d}_h^k,\mathbf{W}_h^{\rm nc}\Lambda^k)\right|=\mathscr{O}(h)$.

Proof of Theorem 22 It is well known that (c.f., e.g., Arnold et al. (2006)), there exists a constant $C_{k,n}$ such that $\mathsf{pic}(\mathbf{d}^k,\mathbf{W}_h\Lambda^k)\leqslant C_{k,n}$, and, $\mathsf{pic}(\mathbf{d}^k,\mathbf{W}_{h0}\Lambda^k)\leqslant C_{k,n}$, which implies immediately that $\mathsf{pic}(\delta_{k+1},\mathbf{W}_{h0}^*\Lambda^{k+1})$ and $\mathsf{pic}(\delta_{k+1},\mathbf{W}_h^*\Lambda^{k+1})$ are uniformly bounded. It follows then $\mathsf{pic}(\mathbf{d}_h^k,\mathbf{W}_h^{\rm nc}\Lambda^k)\leqslant C_{k,n}$. □

Similarly, $\mathsf{pic}(\mathbf{d}_{h0}^k,\mathbf{W}_h^{\rm nc}\Lambda^k)\leqslant C_{k,n}$.

3.1.3. Finite element schemes for elliptic variational problems

Consider the elliptic variational problem: given $\mathbf{f}\in L^2\Lambda^k$, find $\omega\in H\Lambda^k$, such that

$$\langle\mathbf{d}^k\omega,\mathbf{d}^k\mu\rangle_{L^2\Lambda^{k+1}}+\langle\omega,\mu\rangle_{L^2\Lambda^k}=\langle\mathbf{f},\mu\rangle_{L^2\Lambda^k},\ \ \forall\,\mu\in L^2\Lambda^k.\tag{3.5}$$

It follows that $\mathbf{d}^k\omega\in H_0^*\Lambda^{k+1}$, and $\delta_{k+1}\mathbf{d}^k\omega+\omega=\mathbf{f}$.

We consider its finite element discretization: find $\omega\in\mathbf{W}_h^{\rm nc}\Lambda^k$, such that

$$\langle\mathbf{d}_h^k\omega_h,\mathbf{d}_h^k\mu_h\rangle_{L^2\Lambda^{k+1}}+\langle\omega_h,\mu_h\rangle_{L^2\Lambda^k}=\langle\mathbf{f},\mu_h\rangle_{L^2\Lambda^k},\ \ \forall\mu_h\in\mathbf{W}_h^{\rm nc}\Lambda^k. \tag{3.6}$$

Immediately (3.5) and (3.6) are well-posed.

Theorem 26 *Let ω and ω_h be the solutions of* (3.5) *and* (3.6)*, respectively.*

$$\|\omega-\omega_h\|_{\mathbf{d}_h^k}\leqslant 2\inf_{\mu_h\in\mathbf{W}_h^{\rm nc}}\|\omega-\mu_h\|_{\mathbf{d}_h^k}+\inf_{\tau_h\in\mathbf{W}_{h0}^*\Lambda^{k+1}}\|\mathbf{d}^k\omega-\tau_h\|_{\delta_{k+1}}.$$

The proof is the same as that of Theorem 14, and is omitted here.

3.2. *Discrete Helmholtz-Hodge decompositions of $\mathscr{P}_0\Lambda^k(\mathscr{G}_h)$*

Theorem 27 (Discrete Helmholtz decomposition) *Orthogonal in $L^2\Lambda^k(\Omega)$, for $1\leqslant k\leqslant n$,*

$$\mathscr{P}_0\Lambda^k(\mathscr{G}_h)=\mathscr{R}(\mathbf{d}_h^{k-1},\mathbf{W}_h^{\rm nc}\Lambda^{k-1})\oplus^\perp\mathscr{N}(\delta_k,\mathbf{W}_{h0}^*\Lambda^k)=\mathscr{R}(\mathbf{d}_h^{k-1},\mathbf{W}_{h0}^{\rm nc}\Lambda^{k-1})\oplus^\perp\mathscr{N}(\delta_k,\mathbf{W}_h^*\Lambda^k);$$

for $0\leqslant k\leqslant n-1$,

$$\mathscr{P}_0\Lambda^k(\mathscr{G}_h)=\mathscr{N}(\mathbf{d}_h^k,\mathbf{W}_h^{\rm nc}\Lambda^k)\oplus^\perp\mathscr{R}(\delta_{k+1},\mathbf{W}_{h0}^*\Lambda^{k+1})=\mathscr{N}(\mathbf{d}_h^k,\mathbf{W}_{h0}^{\rm nc}\Lambda^k)\oplus^\perp\mathscr{R}(\delta_{k+1},\mathbf{W}_h^*\Lambda^{k+1}).$$

Proof We are going to show, for $1\leqslant k\leqslant n$,

$$\mathscr{P}_0\Lambda^k(\mathscr{G}_h)=\mathscr{R}(\mathbf{d}_h^{k-1},\mathbf{W}_h^{\rm nc}\Lambda^{k-1})\oplus^\perp\mathscr{N}(\delta_k,\mathbf{W}_{h0}^*\Lambda^k),$$

and other assertions follow the same way.

By construction, $\mathscr{P}_0\Lambda^k(\mathscr{G}_h)$ contains $\mathscr{R}(\mathbf{d}_h^{k-1},\mathbf{W}_h^{\rm nc}\Lambda^{k-1})\oplus^\perp\mathscr{N}(\delta_k,\mathbf{W}_{h0}^*\Lambda^k)$. Conversely, let $\sigma_h\in\mathscr{P}_0\Lambda^k(\mathscr{G}_h)\ominus^\perp\mathscr{R}(\mathbf{d}_h^{k-1},\mathbf{W}_h^{\rm nc}\Lambda^{k-1})$. Then for any $\mu_h\in\mathbf{W}_h^{\rm nc}\Lambda^{k-1}$,

$$\sum_{T\in\mathscr{G}_h}\langle\sigma_h,\mathbf{d}^{k-1}\mu_h\rangle_{\mathbf{L}^2\Lambda^k(T)}+\langle\delta_k\sigma_h,\mu_h\rangle_{\mathbf{L}^2\Lambda^{k-1}(T)}=\langle\sigma_h,\mathbf{d}_h^{k-1}\mu_h\rangle_{\mathbf{L}^2\Lambda^k}=0.$$

Namely $\sigma_h\in\mathbf{W}_{h0}^*\Lambda^k$ and further $\sigma_h\in\mathscr{N}(\delta_k,\mathbf{W}_{h0}^*\Lambda^k)$. This completes the proof. □

By noting that, for $1\leqslant k\leqslant n-1$,

$$\mathscr{P}_0\Lambda^k(\mathscr{G}_h)=\mathscr{R}(\mathbf{d}_h^{k-1},\mathbf{W}_h^{\rm nc}\Lambda^{k-1})\oplus^\perp\mathscr{N}(\delta_k,\mathbf{W}_{h0}^*\Lambda^k)=\mathscr{N}(\mathbf{d}_h^k,\mathbf{W}_h^{\rm nc}\Lambda^k)\oplus^\perp\mathscr{R}(\delta_{k+1},\mathbf{W}_{h0}^*\Lambda^{k+1}),$$

we have immediately that, for $1\leqslant k\leqslant n-1$,

$$\mathscr{R}(\mathbf{d}_h^{k-1},\mathbf{W}_h^{\rm nc}\Lambda^{k-1})\subset\mathscr{N}(\mathbf{d}_h^k,\mathbf{W}_h^{\rm nc}\Lambda^k)\Longleftrightarrow\mathscr{R}(\delta_{k+1},\mathbf{W}_{h0}^*\Lambda^{k+1})\subset\mathscr{N}(\delta_k,\mathbf{W}_{h0}^*\Lambda^k). \tag{3.7}$$

Further, we can construct the discrete Poincaré-Lefschetz duality identities below.

Theorem 28 (Discrete Poincaré-Lefschetz duality) *For* $1 \leqslant k \leqslant n-1$,

$$\mathcal{N}(\mathbf{d}_h^k, \mathbf{W}_h^{\mathrm{nc}}\Lambda^k) \ominus^{\perp} \mathcal{R}(\mathbf{d}_h^{k-1}, \mathbf{W}_h^{\mathrm{nc}}\Lambda^{k-1}) = \mathcal{N}(\delta_k, \mathbf{W}_{h0}^{*}\Lambda^k) \ominus^{\perp} \mathcal{R}(\delta_{k+1}, \mathbf{W}_{h0}^{*}\Lambda^{k+1})$$

and

$$\mathcal{N}(\mathbf{d}_h^k, \mathbf{W}_{h0}^{\mathrm{nc}}\Lambda^k) \ominus^{\perp} \mathcal{R}(\mathbf{d}_h^{k-1}, \mathbf{W}_{h0}^{\mathrm{nc}}\Lambda^{k-1}) = \mathcal{N}(\delta_k, \mathbf{W}_{h}^{*}\Lambda^k) \ominus^{\perp} \mathcal{R}(\delta_{k+1}, \mathbf{W}_{h}^{*}\Lambda^{k+1}).$$

Denote $\mathfrak{H}_h^{\mathrm{nc}}\Lambda^k := \mathcal{N}(\mathbf{d}_h^k, \mathbf{W}_h^{\mathrm{nc}}\Lambda^k) \ominus^{\perp} \mathcal{R}(\mathbf{d}_h^{k-1}, \mathbf{W}_h^{\mathrm{nc}}\Lambda^{k-1})$ and $\mathfrak{H}_{h0}^{\mathrm{nc}}\Lambda^k := \mathcal{N}(\mathbf{d}_h^k, \mathbf{W}_{h0}^{\mathrm{nc}}\Lambda^k) \ominus^{\perp} \mathcal{R}(\mathbf{d}_h^{k-1}, \mathbf{W}_{h0}^{\mathrm{nc}}\Lambda^{k-1})$. We have the discrete orthogonal decomposition of $\mathcal{P}_0\Lambda^k(\mathcal{G}_h)$ below.

Theorem 29 (Discrete Hodge decomposition) *For* $1 \leqslant k \leqslant n-1$,

$$\begin{aligned}\mathcal{P}_0\Lambda^k(\mathcal{G}_h) &= \mathcal{R}(\mathbf{d}_h^{k-1}, \mathbf{W}_h^{\mathrm{nc}}\Lambda^{k-1}) \oplus^{\perp} \mathfrak{H}_{h0}^{*}\Lambda^k (= \mathfrak{H}_h^{\mathrm{nc}}\Lambda^k) \oplus^{\perp} \mathcal{R}(\delta_{k+1}, \mathbf{W}_{h0}^{*}\Lambda^{k+1}) \\ &= \mathcal{R}(\mathbf{d}_h^{k-1}, \mathbf{W}_{h0}^{\mathrm{nc}}\Lambda^{k-1}) \oplus^{\perp} \mathfrak{H}_{h0}^{\mathrm{nc}}\Lambda^k (= \mathfrak{H}_h^{*}\Lambda^k) \oplus^{\perp} \mathcal{R}(\delta_{k+1}, \mathbf{W}_{h}^{*}\Lambda^{k+1}).\end{aligned} \tag{3.8}$$

Remark 30 *Within classical FEEC theory, as demonstrated in Arnold (2018, (5.6)), discrete Hodge decompositions for $H\Lambda^k$ finite element spaces "V_h^k"(identical to $\mathbf{W}_h\Lambda^k$ in the present paper) are established, allowing for in-contractible domains, which reads $V_h^k = \mathcal{B}_h^k \ominus^{\perp} \mathfrak{H}_h^k \ominus^{\perp} \mathcal{B}_{k,h}^{*}$, where $\mathcal{B}_{k,h}^{*}$ is the range of a* **globally** *defined operator d_{jh}^{*}. According to (3.7), similar decompositions can be rebuilt associated with $\mathbf{W}_h^{\mathrm{nc}}\Lambda^k$. Contrastly, Theorem 29 is of a discretization of $L^2\Lambda^k$ for both contractible and in-contractible domains. Notably, all discrete operators involved are* **locally** *defined, acting cell by cell.*

3.3. *Commutative diagrams*

Lemma 31 *For any $\mu \in H\Lambda^k(T)$, $0 \leqslant k \leqslant n-1$, $\mathbb{I}_T^{\mathbf{d}^{k+1}}\mathbf{d}^k\mu = \mathbf{d}^k\mathbb{I}_T^{\mathbf{d}^k}\mu$.*

Proof Since $\mathbf{d}^{k+1}\mathbf{d}^k\mu = 0$, $\mathbf{d}^{k+1}\mathbb{I}_T^{\mathbf{d}^{k+1}}\mathbf{d}^k\mu = 0$. Further, $\mathbb{I}_T^{\mathbf{d}^{k+1}}\mathbf{d}^k\mu \in \mathcal{P}_0\Lambda^{k+1}$. Then,

$$\begin{aligned}&\langle \mathbb{I}_T^{\mathbf{d}^{k+1}}\mathbf{d}^k\mu, \delta_{k+2}\eta\rangle_{L^2\Lambda^{k+1}(T)} - \underline{\langle \mathbf{d}^{k+1}\mathbb{I}_T^{\mathbf{d}^{k+1}}\mathbf{d}^k\mu, \eta\rangle_{L^2\Lambda^{k+2}(T)}} \\ &= \langle \mathbf{d}^k\mu, \delta_{k+2}\eta\rangle_{L^2\Lambda^{k+1}(T)} - \underline{\langle \mathbf{d}^{k+1}\mathbf{d}^k\mu, \eta\rangle_{L^2\Lambda^{k+2}(T)}} = \langle \mathbf{d}^k\mu, \delta_{k+2}\eta\rangle_{L^2\Lambda^{k+1}(T)} - \underline{\langle \mu, \delta_{k+1}\delta_{k+2}\eta\rangle_{L^2\Lambda^{k}(T)}} \\ &= \langle \mathbf{d}^k\mathbb{I}_T^{\mathbf{d}^k}\mu, \delta_{k+2}\eta\rangle_{L^2\Lambda^{k+1}(T)} - \underline{\langle \mathbb{I}_T^{\mathbf{d}^k}\mu, \delta_{k+1}\delta_{k+2}\eta\rangle_{L^2\Lambda^{k}(T)}}, \quad \forall \eta \in \mathcal{P}_1^{*,-}\Lambda^{k+2}(T).\end{aligned}$$

Here we use underline to label the vanishing terms. Therefore, $\mathbb{I}_T^{\mathbf{d}^{k+1}}\mathbf{d}^k\mu = \mathbf{d}^k\mathbb{I}_T^{\mathbf{d}^k}\mu$. $\square$

Immediately we have, for any $\mu \in H\Lambda^k(\Omega)$, $\mathbb{I}_h^{\mathbf{d}^{k+1}}\mathbf{d}^k\mu = \mathbf{d}_h^k\mathbb{I}_h^{\mathbf{d}^k}\mu$, $0 \leqslant k \leqslant n-1$.
We summarize all above to theorem below.

Theorem 32 *The following de Rham complexes commute:*

$$\begin{array}{ccccccccccc} \mathbb{R} & \xrightarrow{inc} & H\Lambda^0 & \xrightarrow{\mathbf{d}^0} & H\Lambda^1 & \xrightarrow{\mathbf{d}^1} & \dots & \xrightarrow{\mathbf{d}^{n-1}} & H\Lambda^n \\ & & \downarrow \mathbb{I}_h^{\mathbf{d}^0} & & \downarrow \mathbb{I}_h^{\mathbf{d}^1} & & & & \downarrow \mathbb{I}_h^{\mathbf{d}^n} \\ \mathbb{R} & \xrightarrow{inc} & \mathbf{W}_h^{\mathrm{nc}}\Lambda^0 & \xrightarrow{\mathbf{d}_h^0} & \mathbf{W}_h^{\mathrm{nc}}\Lambda^1 & \xrightarrow{\mathbf{d}_h^1} & \dots & \xrightarrow{\mathbf{d}_h^{n-1}} & \mathbf{W}_h^{\mathrm{nc}}\Lambda^n \end{array} ; \qquad (3.9)$$

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H_0\Lambda^0 & \xrightarrow{\mathbf{d}^0} & H_0\Lambda^1 & \xrightarrow{\mathbf{d}^1} & \dots & \xrightarrow{\mathbf{d}^{n-1}} & H_0\Lambda^n \\ & & \downarrow \mathbb{I}_h^{\mathbf{d}^0} & & \downarrow \mathbb{I}_h^{\mathbf{d}^1} & & & & \downarrow \mathbb{I}_h^{\mathbf{d}^n} \\ 0 & \longrightarrow & \mathbf{W}_{h0}^{\mathrm{nc}}\Lambda^0 & \xrightarrow{\mathbf{d}_h^0} & \mathbf{W}_{h0}^{\mathrm{nc}}\Lambda^1 & \xrightarrow{\mathbf{d}_h^1} & \dots & \xrightarrow{\mathbf{d}_h^{n-1}} & \mathbf{W}_{h0}^{\mathrm{nc}}\Lambda^n \end{array} . \qquad (3.10)$$

Remark 33 *Given Theorem 28 the discrete Poincaré-Lefschetz duality, we are actually led to that, once one of the four complexes in (3.9) and (3.10) is exact, so are the three others.*

4. Discretization of the Hodge Laplace problem with nonconforming finite element spaces

In this section, we study the discretizations of the Hodge Laplace problem: given $\mathbf{f}\in L^2\Lambda^k$, with $\mathbf{P}^k_{\mathfrak{H}}$ the L^2 projection to $\mathfrak{H}\Lambda^k$, find $\omega\in H\Lambda^k(\Omega)\cap H_0^*\Lambda^k(\Omega)$ with $\mathbf{d}^k\omega\in H_0^*\Lambda^{k+1}(\Omega)$, such that

$$\omega\perp\mathfrak{H}\Lambda^k(\Omega),\quad \text{and}\quad \delta_{k+1}\mathbf{d}^k\omega+\mathbf{d}^{k-1}\delta_k\omega=\mathbf{f}-\mathbf{P}^k_{\mathfrak{H}}\mathbf{f}. \qquad (4.1)$$

The primal weak formulation is: find $\omega\in H\Lambda^k\cap H_0^*\Lambda^k$, such that

$$\left\{\begin{array}{rll} \langle\omega,\varsigma\rangle_{L^2\Lambda^k} & =0, & \forall\varsigma\in\mathfrak{H}\Lambda^k, \\ \langle\mathbf{d}^k\omega,\mathbf{d}^k\mu\rangle_{L^2\Lambda^{k+1}}+\langle\delta_k\omega,\delta_k\mu\rangle_{L^2\Lambda^{k-1}} & =\langle\mathbf{f}-\mathbf{P}^k_{\mathfrak{H}}\mathbf{f},\mu\rangle_{L^2\Lambda^k}, & \forall\mu\in H\Lambda^k(\Omega)\cap H_0^*\Lambda^k(\Omega). \end{array}\right. \qquad (4.2)$$

A standard mixed formulation based on $\omega\in H\Lambda^k$ is generally used (Arnold, 2018), which seeks $(\omega^{\mathbf{p}},\sigma^{\mathbf{p}},\vartheta^{\mathbf{p}})\in H\Lambda^k\times H\Lambda^{k-1}\times\mathfrak{H}\Lambda^k$, such that, for $(\mu,\tau,\varsigma)\in H\Lambda^k\times H\Lambda^{k-1}\times\mathfrak{H}\Lambda^k$,

$$\left\{\begin{array}{llll} & & \langle\omega^{\mathbf{p}},\varsigma\rangle_{L^2\Lambda^k} & =0 \\ & \langle\sigma^{\mathbf{p}},\tau\rangle_{L^2\Lambda^{k+1}} & -\langle\omega^{\mathbf{p}},\mathbf{d}^{k-1}\tau\rangle_{L^2\Lambda^k} & =0 \\ \langle\vartheta^{\mathbf{p}},\mu\rangle_{L^2\Lambda^k} & +\langle\mathbf{d}^{k-1}\sigma^{\mathbf{p}},\mu\rangle_{L^2\Lambda^k} & +\langle\mathbf{d}^k\omega^{\mathbf{p}},\mathbf{d}^k\mu\rangle_{L^2\Lambda^{k-1}} & =\langle\mathbf{f},\mu\rangle_{L^2\Lambda^k} \end{array}\right. . \qquad (4.3)$$

In this section, we investigate the application of the nonconforming finite element spaces to the discretizations of this classical formulation and to a new "completely" mixed formulation.

Remark 34 *Here we call (4.3) "primal" mixed formulation, and use the supscript* $^{\mathbf{p}}$ *to label that. Actually, for a function in* $H\Lambda^k\cap H_0^*\Lambda^k$, *(4.3) sets* $\omega^{\mathbf{p}}\in H\Lambda^k$, *and impose the continuity that it also belongs to* $H_0^*\Lambda^k$ *in a dual way. It is natural to set* $\omega\in H_0^*\Lambda^k$ *and to impose the continuity that it also belongs to* $H\Lambda^k$ *in a dual way. Namely, we are to set an auxiliary mixed formulation, which seeks* $(\omega^{\mathbf{d}},\zeta^{\mathbf{d}},\vartheta^{\mathbf{d}})\in H_0^*\Lambda^k\times H_0^*\Lambda^{k+1}\times\mathfrak{H}_0^*\Lambda^k$, *such that, for* $(\mu,\eta,\varsigma)\in H_0^*\Lambda^k\times H_0^*\Lambda^{k+1}\times\mathfrak{H}_0^*\Lambda^k$,

$$\left\{\begin{array}{llll} & & \langle\omega^{\mathbf{d}},\varsigma\rangle_{L^2\Lambda^k} & =0 \\ & \langle\zeta^{\mathbf{d}},\eta\rangle_{L^2\Lambda^{k+1}} & -\langle\omega^{\mathbf{d}},\delta_{k+1}\eta\rangle_{L^2\Lambda^k} & =0 \\ \langle\vartheta^{\mathbf{d}},\mu\rangle_{L^2\Lambda^k} & +\langle\delta_{k+1}\zeta^{\mathbf{d}},\mu\rangle_{L^2\Lambda^k} & +\langle\delta_k\omega^{\mathbf{d}},\delta_k\mu\rangle_{L^2\Lambda^{k-1}} & =\langle\mathbf{f},\mu\rangle_{L^2\Lambda^k} \end{array}\right. . \qquad (4.4)$$

In the aforementioned sense, this mixed formulation can be viewed as a dual one of (4.3).

Conforming finite elements have been used for discretization of (4.3)*; they are naturally used for* (4.4)*. For example, we can consider the discretization for* (4.4)*: to find* $(\omega_h^{\mathbf{d}},\zeta_h^{\mathbf{d}},\vartheta_h^{\mathbf{d}})\in\mathbf{W}_{h0}^*\Lambda^k\times\mathbf{W}_{h0}^*\Lambda^{k+1}\times\mathfrak{H}_{h0}^*\Lambda^k$*, such that, for* $(\mu_h,\eta_h,\varsigma_h)\in\mathbf{W}_{h0}^*\Lambda^k\times\mathbf{W}_{h0}^*\Lambda^{k+1}\times\mathfrak{H}_{h0}^*\Lambda^k$*,*

$$\left\{\begin{array}{lllll} & & \langle\omega^{\mathbf{d}},\varsigma\rangle_{L^2\Lambda^k} & =0 \\ & \langle\mathbf{P}_h^{k+1}\zeta_h^{\mathbf{d}},\mathbf{P}_h^{k+1}\eta_h\rangle_{L^2\Lambda^{k+1}} & -\langle\omega_h^{\mathbf{d}},\delta_{k+1}\eta_h\rangle_{L^2\Lambda^k} & =0 \\ \langle\vartheta_h^{\mathbf{d}},\mu_h\rangle_{L^2\Lambda^k} & +\langle\delta_{k+1}\zeta_h^{\mathbf{d}},\mu_h\rangle_{L^2\Lambda^k} & +\langle\delta_k\omega_h^{\mathbf{d}},\delta_k\mu_h\rangle_{L^2\Lambda^{k-1}} & =\langle\mathbf{f},\mathbf{P}_h^k\mu_h\rangle_{L^2\Lambda^k}\end{array}\right. . \tag{4.5}$$

The well-posedness of (4.5) *is the same as that of* (4.8) *below. The convergence analysis of* (4.5) *can be done in a classical way; precisely, denote by* $(\bar\omega_h^{\mathbf{d}},\bar\zeta_h^{\mathbf{d}},\bar\vartheta_h^{\mathbf{d}})\in\mathbf{W}_{h0}^*\Lambda^k\times\mathbf{W}_{h0}^*\Lambda^{k+1}\times\mathfrak{H}_{h0}^*\Lambda^k$ *and* $(\tilde\omega_h^{\mathbf{d}},\tilde\zeta_h^{\mathbf{d}},\tilde\vartheta_h^{\mathbf{d}})\in\mathbf{W}_{h0}^*\Lambda^k\times\mathbf{W}_{h0}^*\Lambda^{k+1}\times\mathfrak{H}_{h0}^*\Lambda^k$ *the respective solutions of the auxiliary problems*

$$\left\{\begin{array}{llll} & & \langle\bar\omega^{\mathbf{d}},\varsigma\rangle_{L^2\Lambda^k} & =0 \\ & \langle\bar\zeta_h^{\mathbf{d}},\eta_h\rangle_{L^2\Lambda^{k+1}} & -\langle\bar\omega_h^{\mathbf{d}},\delta_{k+1}\eta_h\rangle_{L^2\Lambda^k} & =0 \\ \langle\bar\vartheta_h^{\mathbf{d}},\mu_h\rangle_{L^2\Lambda^k} & +\langle\delta_{k+1}\bar\zeta_h^{\mathbf{d}},\mu_h\rangle_{L^2\Lambda^k} & +\langle\delta_k\bar\omega_h^{\mathbf{d}},\delta_k\mu_h\rangle_{L^2\Lambda^{k-1}} & =\langle\mathbf{P}_h^k\mathbf{f},\mu_h\rangle_{L^2\Lambda^k}\end{array}\right. , \tag{4.6}$$

and

$$\left\{\begin{array}{llll} & & \langle\tilde\omega^{\mathbf{d}},\varsigma\rangle_{L^2\Lambda^k} & =0 \\ & \langle\tilde\zeta_h^{\mathbf{d}},\eta_h\rangle_{L^2\Lambda^{k+1}} & -\langle\tilde\omega_h^{\mathbf{d}},\delta_{k+1}\eta_h\rangle_{L^2\Lambda^k} & =0 \\ \langle\tilde\vartheta_h^{\mathbf{d}},\mu_h\rangle_{L^2\Lambda^k} & +\langle\delta_{k+1}\tilde\zeta_h^{\mathbf{d}},\mu_h\rangle_{L^2\Lambda^k} & +\langle\delta_k\tilde\omega_h^{\mathbf{d}},\delta_k\mu_h\rangle_{L^2\Lambda^{k-1}} & =\langle\mathbf{f},\mu_h\rangle_{L^2\Lambda^k}\end{array}\right. . \tag{4.7}$$

It follows by standard procedure that

$$\|(\omega_h^{\mathbf{d}},\zeta_h^{\mathbf{d}},\vartheta_h^{\mathbf{d}})-(\bar\omega_h^{\mathbf{d}},\bar\zeta_h^{\mathbf{d}},\bar\vartheta_h^{\mathbf{d}})\|_{H^*\Lambda^k\times H^*\Lambda^{k+1}\times L^2\Lambda^k}\leqslant Ch\|\mathbf{P}_h^k\mathbf{f}\|_{L^2\Lambda^k}\leqslant Ch\|\mathbf{f}\|_{L^2\Lambda^k},$$

and

$$\|(\bar\omega_h^{\mathbf{d}},\bar\zeta_h^{\mathbf{d}},\bar\vartheta_h^{\mathbf{d}})-(\tilde\omega_h^{\mathbf{d}},\tilde\zeta_h^{\mathbf{d}},\tilde\vartheta_h^{\mathbf{d}})\|_{H^*\Lambda^k\times H^*\Lambda^{k+1}\times L^2\Lambda^k}\leqslant Ch\|\mathbf{f}\|_{L^2\Lambda^k}.$$

Meanwhile, the classical analysis (cf. Arnold et al. (2006, Theorem 7.10 and its proof)) holds as

$$\|(\omega^{\mathbf{d}},\zeta^{\mathbf{d}},\vartheta^{\mathbf{d}})-(\tilde\omega_h^{\mathbf{d}},\tilde\zeta_h^{\mathbf{d}},\tilde\vartheta_h^{\mathbf{d}})\|_{H^*\Lambda^k\times H^*\Lambda^{k+1}\times L^2\Lambda^k}\leqslant Ch^s\|\mathbf{f}\|_{L^2\Lambda^k},$$

if the domain Ω *is s-regular. The convergence analysis of* (4.5) *then follows.*

4.1. *Nonconforming discretization of* (4.3)

By the newly designed nonconforming finite element spaces, the discrete problem is: to find $(\omega_h^{\mathbf{p}},\sigma_h^{\mathbf{p}},\vartheta_h^{\mathbf{p}})\in\mathbf{W}_h^{\rm nc}\Lambda^k\times\mathbf{W}_h^{\rm nc}\Lambda^{k-1}\times\mathfrak{H}_h^{\rm nc}\Lambda^k$, such that, for $(\mu_h,\tau_h,\varsigma_h)\in\mathbf{W}_h^{\rm nc}\Lambda^k\times\mathbf{W}_h^{\rm nc}\Lambda^{k-1}\times\mathfrak{H}_h^{\rm nc}\Lambda^k$,

$$\left\{\begin{array}{llll} & & \langle\omega_h^{\mathbf{p}},\varsigma_h\rangle_{L^2\Lambda^k} & =0 \\ & \langle\mathbf{P}_h^{k-1}\sigma_h^{\mathbf{p}},\mathbf{P}_h^{k-1}\tau_h\rangle_{L^2\Lambda^{k+1}} & -\langle\omega_h^{\mathbf{p}},\mathbf{d}_h^{k-1}\tau_h\rangle_{L^2\Lambda^k} & =0 \\ \langle\vartheta_h^{\mathbf{p}},\mu_h\rangle_{L^2\Lambda^k} & +\langle\mathbf{d}_h^{k-1}\sigma_h^{\mathbf{p}},\mu_h\rangle_{L^2\Lambda^k} & +\langle\mathbf{d}_h^k\omega_h^{\mathbf{p}},\mathbf{d}_h^k\mu_h\rangle_{L^2\Lambda^{k-1}} & =\langle\mathbf{f},\mathbf{P}_h^k\mu_h\rangle_{L^2\Lambda^k}\end{array}\right. . \tag{4.8}$$

To verify the well-posedness of (4.8), following (Arnold, 2018, Section 4.2.2), writing $X_h:=\mathbf{W}_h^{\rm nc}\Lambda^k\times\mathbf{W}_h^{\rm nc}\Lambda^{k-1}\times\mathfrak{H}_h^{\rm nc}\Lambda^k$, with $\|(\mu_h,\tau_h,\varsigma_h)\|_{X_h}:=\|\mu_h\|_{\mathbf{d}_h^k}+\|\tau_h\|_{\mathbf{d}_h^{k-1}}+\|\varsigma_h\|_{L^2\Lambda^k}$, denoting on $X_h\times X_h$

$$\begin{aligned}B_h((\omega_h,\sigma_h,\vartheta_h),(\mu_h,\tau_h,\varsigma_h)):=&\langle\mathbf{P}_h^{k-1}\sigma_h^{\mathbf{p}},\mathbf{P}_h^{k-1}\tau_h\rangle_{L^2\Lambda^{k+1}}-\langle\omega_h^{\mathbf{p}},\mathbf{d}_h^{k-1}\tau_h\rangle_{L^2\Lambda^k}\\&-\langle\vartheta_h^{\mathbf{p}},\mu_h\rangle_{L^2\Lambda^k}-\langle\mathbf{d}_h^{k-1}\sigma_h^{\mathbf{p}},\mu_h\rangle_{L^2\Lambda^k}-\langle\mathbf{d}_h^k\omega_h^{\mathbf{p}},\mathbf{d}_h^k\mu_h\rangle_{L^2\Lambda^{k-1}}-\langle\omega_h^{\mathbf{p}},\varsigma_h\rangle_{L^2\Lambda^k},\end{aligned} \tag{4.9}$$

we show the uniform inf-sup condition that

$$\inf_{0\neq(\omega_h,\sigma_h,\vartheta_h)\in X_h}\ \sup_{0\neq(\mu_h,\tau_h,\varsigma_h)\in X_h}\frac{B_h((\omega_h,\sigma_h,\vartheta_h),(\mu_h,\tau_h,\varsigma_h))}{\|(\omega_h,\sigma_h,\vartheta_h)\|_{X_h}\|(\mu_h,\tau_h,\varsigma_h)\|_{X_h}}\geqslant\gamma>0. \tag{4.10}$$

Given $(\omega_h,\sigma_h,\vartheta_h)\in X_h$, we can decompose orthogonally $\omega_h=\mathbf{d}_h^{k-1}\rho_h+\omega_h^{\mathfrak{H}}+\omega_h^{\lrcorner}$, with $\rho_h\in\mathbf{W}_h^{\rm nc}\Lambda^{k-1}$, $\omega_h^{\mathfrak{H}}\in\mathfrak{H}_h^{\rm nc}\Lambda^k$, and $\omega_h^{\lrcorner}$ orthogonal to $\mathscr{N}(\mathbf{d}_h^k,\mathbf{W}_h^{\rm nc}\Lambda^k)$, such that, by the discrete Poincaré inequality (Theorem 22), $\|\rho_h\|_{\mathbf{d}_h^{k-1}}\leqslant c_P\|\mathbf{d}_h^{k-1}\rho_h\|_{L^2\Lambda^k}$.

Now, set $\tau_h=\sigma_h-\frac{1}{c_P^2}\rho_h$, $\mu_h=-\omega_h-\mathbf{d}_h^{k-1}\sigma_h-\vartheta_h$, and $\varsigma_h=-\vartheta_h+\omega_h^{\mathfrak{H}}$, then

$$\|(\mu_h,\tau_h,\varsigma_h)\|_{X_h}\leqslant C\|(\omega_h,\sigma_h,\vartheta_h)\|_{X_h},$$

and

$$\begin{aligned}B_h((\omega_h,\sigma_h,\vartheta_h),(\mu_h,\tau_h,\varsigma_h))&=\|\mathbf{P}_h^{k-1}\sigma_h\|^2_{L^2\Lambda^{k-1}}+\|\mathbf{d}_h^{k-1}\sigma_h\|^2_{L^2\Lambda^k}+\|\mathbf{d}_h^k\omega_h\|^2_{L^2\Lambda^{k+1}}\\&\quad+\|\vartheta_h\|^2_{L^2\Lambda^k}+\|\omega_h^{\mathfrak{H}}\|^2_{L^2\Lambda^k}+\frac{1}{c_P^2}\|\mathbf{d}_h^{k-1}\rho_h\|_{L^2\Lambda^k}-\frac{1}{c_P^2}\langle\sigma_h,\rho_h\rangle_{L^2\Lambda^{k-1}}.\end{aligned}$$

Note further that

$$\begin{aligned}\langle\sigma_h,\rho_h\rangle_{L^2\Lambda^{k-1}}&\leqslant\|\sigma_h\|_{L^2\Lambda^{k-1}}\|\rho_h\|_{L^2\Lambda^{k-1}}\\&\leqslant\frac{c_P^2}{2}\|\sigma_h\|^2_{L^2\Lambda^{k-1}}+\frac{1}{2c_P^2}\|\rho_h\|^2_{L^2\Lambda^{k-1}}\leqslant\frac{c_P^2}{2}\|\sigma_h\|^2_{L^2\Lambda^{k-1}}+\frac{1}{2}\|\mathbf{d}_h^{k-1}\rho_h\|^2_{L^2\Lambda^k}\end{aligned}$$

Thus

$$\begin{aligned}B_h((\omega_h,\sigma_h,\vartheta_h),(\mu_h,\tau_h,\varsigma_h))&\geqslant\|\mathbf{P}_h^{k-1}\sigma_h\|^2_{L^2\Lambda^{k-1}}+\|\mathbf{d}_h^{k-1}\sigma_h\|^2_{L^2\Lambda^k}-\frac{1}{2}\|\sigma_h\|^2_{L^2\Lambda^{k-1}}+\|\mathbf{d}_h^k\omega_h\|^2_{L^2\Lambda^{k+1}}\\&\quad+\|\vartheta_h\|^2_{L^2\Lambda^k}+\|\omega_h^{\mathfrak{H}}\|^2_{L^2\Lambda^k}+\frac{1}{c_P^2}\|\mathbf{d}_h^{k-1}\rho_h\|_{L^2\Lambda^k}\\&\geqslant\frac{1}{2}\|\sigma_h\|^2_{L^2\Lambda^{k-1}}+(1-Ch^2)\|\mathbf{d}_h^{k-1}\sigma_h\|^2_{L^2\Lambda^k}+\|\mathbf{d}_h^k\omega_h\|^2_{L^2\Lambda^{k+1}}\\&\quad+\|\vartheta_h\|^2_{L^2\Lambda^k}+\|\omega_h^{\mathfrak{H}}\|^2_{L^2\Lambda^k}+\frac{1}{c_P^2}\|\mathbf{d}_h^{k-1}\rho_h\|_{L^2\Lambda^k}.\end{aligned}$$

Note that $\mathbf{d}_h^k\omega_h^{\lrcorner}=\mathbf{d}_h^k\omega_h$, and, by Theorem 22, $\|\omega_h^{\lrcorner}\|_{L^2\Lambda^k}\leqslant C\|\mathbf{d}_h^k\omega_h\|_{L^2\Lambda^{k+1}}$. It follows then

$$B_h((\omega_h,\sigma_h,\vartheta_h),(\mu_h,\tau_h,\varsigma_h))\geqslant C\|(\omega_h,\sigma_h,\vartheta_h)\|^2_{X_h},$$

with C depending on the Poincaré inequality only. The inf-sup condition (4.10) is then proved and the well-posedness of (4.8) is verified.

4.2. *A novel mixed element scheme*

It is natural to consider an approach where both $\mathbf{d}^k$ and δ_k are operated in a dual way, and we begin with this "completely" mixed formulation: to find $(\omega^{\rm c},\zeta^{\rm c},\sigma^{\rm c},\vartheta^{\rm c})\in L^2\Lambda^k\times H^*_0\Lambda^{k+1}\times H\Lambda^{k-1}\times\mathfrak{H}\Lambda^k$, such that, for $(\mu,\eta,\tau,\varsigma)\in L^2\Lambda^k\times H^*_0\Lambda^{k+1}\times H\Lambda^{k-1}\times\mathfrak{H}\Lambda^k$,

$$\left\{\begin{array}{lllll} & & & \langle\omega^{\rm c},\varsigma\rangle_{L^2\Lambda^k} & =0\\ & \langle\zeta^{\rm c},\eta\rangle_{L^2\Lambda^{k+1}} & & -\langle\omega^{\rm c},\delta_{k+1}\eta\rangle_{L^2\Lambda^k} & =0\\ & & \langle\sigma^{\rm c},\tau\rangle_{L^2\Lambda^{k-1}} & -\langle\omega^{\rm c},\mathbf{d}^{k-1}\tau\rangle_{L^2\Lambda^k} & =0\\ \langle\vartheta^{\rm c},\mu\rangle_{L^2\Lambda^k} & +\langle\delta_{k+1}\zeta^{\rm c},\mu\rangle_{L^2\Lambda^k} & +\langle\mathbf{d}^{k-1}\sigma^{\rm c},\mu\rangle_{L^2\Lambda^k} & & =\langle\mathbf{f},\mu\rangle_{L^2\Lambda^k}\end{array}\right. . \tag{4.11}$$

Lemma 35 *For $\mathbf{f}\in L^2\Lambda^k$, the problem (4.11) admits a unique solution $(\omega^{\rm c},\zeta^{\rm c},\sigma^{\rm c},\vartheta^{\rm c})$, and*

$$\|\omega^{\rm c}\|_{L^2\Lambda^k}+\|\zeta^{\rm c}\|_{\delta_{k+1}}+\|\sigma^{\rm c}\|_{\mathbf{d}^{k-1}}+\|\vartheta^{\rm c}\|_{L^2\Lambda^k}\leqslant C\|\mathbf{f}\|_{L^2\Lambda^k}. \tag{4.12}$$

Further, $\zeta^{\rm c}=\mathbf{d}^k\omega^{\rm c}$, $\sigma^{\rm c}=\delta_k\omega^{\rm c}$, and $\omega^{\rm c}$ solves (4.2).

Proof For (4.12), we only have to verify Brezzi's conditions, which hold by the orthogonal Hodge decomposition

$$L^2\Lambda^k=\mathscr{R}(\mathbf{d}^{k-1},H\Lambda^{k-1})\oplus^\perp\mathfrak{H}\Lambda^k\oplus^\perp\mathscr{R}(\delta_{k+1},H^*_0\Lambda^{k+1}),$$

together with the closeness of $\mathscr{R}(\mathbf{d}^{k-1},H\Lambda^{k-1})$ and $\mathscr{R}(\delta_{k+1},H^*_0\Lambda^{k+1})$. The remaining assertions are straightforward. The proof is completed. □

A lowest-degree stable discretization of (4.11) is: find $(\omega^{\rm c}_h,\zeta^{\rm c}_h,\sigma^{\rm c}_h,\vartheta^{\rm c}_h)\in\mathscr{P}_0\Lambda^k(\mathscr{G}_h)\times\mathbf{W}^*_{h0}\Lambda^{k+1}\times\mathbf{W}^{\rm nc}_h\Lambda^{k-1}\times\mathfrak{H}^{\rm nc}_h\Lambda^k$, such that, for $(\mu_h,\eta_h,\tau_h,\varsigma_h)\in\mathscr{P}_0\Lambda^k(\mathscr{G}_h)\times\mathbf{W}^*_{h0}\Lambda^{k+1}\times\mathbf{W}^{\rm nc}_h\Lambda^{k-1}\times\mathfrak{H}^{\rm nc}_h\Lambda^k$,

$$\left\{\begin{array}{lllll} & & & \langle\omega^{\rm c}_h,\varsigma_h\rangle_{L^2\Lambda^k} & =0\\ & \langle\mathbf{P}^{k+1}_h\zeta^{\rm c}_h,\mathbf{P}^{k+1}_h\eta_h\rangle_{L^2\Lambda^{k+1}} & & -\langle\omega^{\rm c}_h,\delta_{k+1}\eta_h\rangle_{L^2\Lambda^k} & =0\\ & & \langle\mathbf{P}^{k-1}_h\sigma^{\rm c}_h,\mathbf{P}^{k-1}_h\tau_h\rangle_{L^2\Lambda^{k-1}} & -\langle\omega^{\rm c}_h,\mathbf{d}^{k-1}_h\tau_h\rangle_{L^2\Lambda^k} & =0\\ \langle\vartheta^{\rm c}_h,\mu_h\rangle_{L^2\Lambda^k} & +\langle\delta_{k+1}\zeta^{\rm c}_h,\mu_h\rangle_{L^2\Lambda^k} & +\langle\mathbf{d}^{k-1}_h\sigma^{\rm c}_h,\mu_h\rangle_{L^2\Lambda^k} & & =\langle\mathbf{f},\mu_h\rangle_{L^2\Lambda^k}\end{array}\right. . \tag{4.13}$$

Lemma 36 *Given $\mathbf{f}\in L^2\Lambda^k$, the problem (4.13) admits a unique solution $(\omega^{\rm c}_h,\zeta^{\rm c}_h,\sigma^{\rm c}_h,\vartheta^{\rm c}_h)$, and*

$$\|\omega^{\rm c}_h\|_{L^2\Lambda^k}+\|\zeta^{\rm c}_h\|_{\delta_{k+1}}+\|\sigma^{\rm c}_h\|_{\mathbf{d}^{k-1}_h}+\|\vartheta^{\rm c}_h\|_{L^2\Lambda^k}\leqslant C\|f\|_{L^2\Lambda^k}.$$

The constant C depends on $\mathsf{pic}(\delta_{k+1},\mathbf{W}^*_{h0}\Lambda^{k+1})$ *and* $\mathsf{pic}(\mathbf{d}^{k-1}_h,\mathbf{W}^{\rm nc}_h\Lambda^{k-1})$.

Again, for the well-posedness of (4.13), we only have to verify Brezzi's conditions, which holds by the discrete Hodge decomposition (3.8). The stable decompositions (3.8) comes true by the aid of the nonconforming space $\mathbf{W}^{\rm nc}_h\Lambda^k$. Hence (4.13) is a new scheme hinted in nonconforming finite element exterior calculus.

4.3. *Equivalences among lowest-degree mixed element schemes*

Lemma 37 *Let* $(\omega_h^{\rm c},\zeta_h^{\rm c},\sigma_h^{\rm c},\vartheta_h^{\rm c})$, $(\omega_h^{\mathbf{p}},\sigma_h^{\mathbf{p}},\vartheta_h^{\mathbf{p}})$ *and* $(\omega_h^{\mathbf{d}},\zeta_h^{\mathbf{d}},\vartheta_h^{\mathbf{d}})$ *be the solutions of* (4.13), (4.8) *and* (4.5)*, respectively. Then*

$$\vartheta_h^{\mathbf{d}}=\vartheta_h^{\rm c},\ \zeta_h^{\mathbf{d}}=\zeta_h^{\rm c},\ \mathbf{P}_h^k\omega_h^{\mathbf{d}}=\omega_h^{\rm c},\ \delta_k\omega_h^{\mathbf{d}}=\mathbf{P}_h^{k-1}\sigma_h^{\rm c},\ \delta_{k+1}\zeta_h^{\mathbf{d}}=\mathbf{P}_h^k\mathbf{f}-\mathbf{d}_h^{k-1}\sigma_h^{\rm c}-\vartheta_h^{\rm c},\tag{4.14}$$

$$\vartheta_h^{\mathbf{p}}=\vartheta_h^{\rm c},\ \sigma_h^{\mathbf{p}}=\sigma_h^{\rm c},\ \mathbf{P}_h^k\omega_h^{\mathbf{p}}=\omega_h^{\rm c},\ \mathbf{d}_h^k\omega_h^{\mathbf{p}}=\mathbf{P}_h^{k+1}\zeta_h^{\rm c},\ \mathbf{d}_h^{k-1}\sigma_h^{\mathbf{p}}=\mathbf{P}_h^k\mathbf{f}-\delta_{k+1}\zeta_h^{\rm c}-\vartheta_h^{\rm c},\tag{4.15}$$

$$\vartheta_h^{\mathbf{d}}=\vartheta_h^{\mathbf{p}},\ \mathbf{P}_h^{k+1}\zeta_h^{\mathbf{d}}=\mathbf{d}_h^k\omega_h^{\mathbf{p}},\ \mathbf{P}_h^k\omega_h^{\mathbf{d}}=\mathbf{P}_h^k\omega_h^{\mathbf{p}},\ \delta_k\omega_h^{\mathbf{d}}=\mathbf{P}_h^{k-1}\sigma_h^{\mathbf{p}},\ \delta_{k+1}\zeta_h^{\mathbf{d}}+\mathbf{d}_h^{k-1}\sigma_h^{\mathbf{p}}=\mathbf{P}_h^k\mathbf{f}-\vartheta_h^{\rm c}.\tag{4.16}$$

Proof Let $(\omega_h^{\mathbf{d}},\zeta_h^{\mathbf{d}},\vartheta_h^{\mathbf{d}})$ be the solution of (4.5). Then, with a $\overline{\sigma}_h\in\mathbf{W}_h^{\rm nc}\Lambda^{k-1}$,

$$\langle\vartheta_h^{\mathbf{d}},\mu_h\rangle_{L^2\Lambda^k}+\langle\delta_{k+1}\zeta_h^{\mathbf{d}},\mu_h\rangle_{L^2\Lambda^k}+\langle\delta_k\omega_h^{\mathbf{d}},\delta_k\mu_h\rangle_{L^2\Lambda^{k-1}}$$
$$+\langle\mathbf{d}_h^{k-1}\overline{\sigma}_h,\mu_h\rangle_{L^2\Lambda^k}-\langle\overline{\sigma}_h,\delta_k\mu_h\rangle_{L^2\Lambda^{k-1}}=\langle\mathbf{f},\mathbf{P}_h^k\mu_h\rangle_{L^2\Lambda^k},$$

for any $\mu_h\in\mathscr{P}_1^{*,-}\Lambda^k(\mathscr{G}_h)$. Choosing arbitrarily $\mu_h\in\mathscr{P}_0\Lambda^k(\mathscr{G}_h)$, we have

$$\vartheta_h^{\mathbf{d}}+\delta_{k+1}\zeta_h^{\mathbf{d}}+\mathbf{d}_h^{k-1}\overline{\sigma}_h=\mathbf{P}_h^k\mathbf{f},\tag{4.17}$$

and

$$\langle\delta_k\omega_h^{\mathbf{d}},\delta_k\mu_h\rangle_{L^2\Lambda^{k-1}}-\langle\overline{\sigma}_h,\delta_k\mu_h\rangle_{L^2\Lambda^{k-1}}=0,\ \ \forall\,\mu_h\in\mathscr{P}_1^{*,-}\Lambda^k(\mathscr{G}_h),$$

which leads to that $\delta_k\omega_h^{\mathbf{d}}=\mathbf{P}_h^{k-1}\overline{\sigma}_h$. Further, noting that $\langle\delta_k\omega_h^{\mathbf{d}},\tau_h\rangle_{L^2\Lambda^{k-1}}=\langle\omega_h^{\mathbf{d}},\mathbf{d}_h^k\tau_h\rangle_{L^2\Lambda^k}$ for $\tau_h\in\mathbf{W}_h^{\rm nc}\Lambda^{k-1}$, we obtain $\langle\mathbf{P}_h^{k-1}\overline{\sigma}_h,\mathbf{P}_h^{k-1}\tau_h^{\mathbf{d}}\rangle_{L^2\Lambda^{k-1}}-\langle\omega_h^{\mathbf{d}},\mathbf{d}_h^{k-1}\tau_h\rangle_{L^2\Lambda^k}=0$ for $\tau_h\in\mathbf{W}_h^{\rm nc}\Lambda^{k-1}$.

In all, $(\mathbf{P}_h^k\omega_h^{\mathbf{d}},\zeta_h^{\mathbf{d}},\overline{\sigma}_h,\vartheta_h^{\mathbf{d}})\in\mathscr{P}_0\Lambda^k(\mathscr{G}_h)\times\mathbf{W}_{h0}^*\Lambda^{k+1}\times\mathbf{W}_h^{\rm nc}\Lambda^{k-1}\times\mathfrak{H}_h^{\rm nc}\Lambda^k$ satisfies the system (4.13), and thus $(\mathbf{P}_h^k\omega_h^{\mathbf{d}},\zeta_h^{\mathbf{d}},\overline{\sigma}_h,\vartheta_h^{\mathbf{d}})=(\omega_h^{\rm c},\zeta_h^{\rm c},\sigma_h^{\rm c},\vartheta_h^{\rm c})$. This proves (4.14). Similarly can (4.15) be proved, and (4.16) follows by (4.14) and (4.15). The proof is completed. □

The convergence analysis of (4.13) and (4.8) follow directly by Remark 34 and Lemma 37, and we omit the details here.

4.4. *A decomposition processes for solving* (4.13)

Firstly, we decomposition (4.13) to two subsystems.

Lemma 38 *Let* $(\omega_h^{\rm c},\zeta_h^{\rm c},\sigma_h^{\rm c},\vartheta_h^{\rm c})$ *be the solution of* (4.13)*, let* ζ_h *and* $\varphi_h\in\mathbf{W}_{h0}^*\Lambda^{k+1}$ *be such that, for any* η_h *and* $\psi_h\in\mathbf{W}_{h0}^*\Lambda^{k+1}$,

$$\left\{\begin{array}{lll}\langle\mathbf{P}_h^{k+1}\zeta_h^{\rm c},\mathbf{P}_h^{k+1}\eta_h\rangle_{L^2\Lambda^{k+1}} & -\langle\delta_{k+1}\varphi_h,\delta_{k+1}\eta_h\rangle_{L^2\Lambda^k} & =0\\ \langle\delta_{k+1}\zeta_h^{\rm c},\delta_{k+1}\psi_h\rangle_{L^2\Lambda^k} & & =\langle\mathbf{f},\delta_{k+1}\psi_h\rangle_{L^2\Lambda^k}\end{array}\right.,\tag{4.18}$$

and let σ_h *and* $\rho_h\in\mathbf{W}_h^{\rm nc}\Lambda^{k-1}$ *be such that, for any* τ_h *and* $\varpi_h\in\mathbf{W}_h^{\rm nc}\Lambda^{k-1}$,

$$\left\{\begin{array}{lll}\langle\mathbf{P}_h^{k-1}\sigma_h^{\rm c},\mathbf{P}_h^{k-1}\tau_h\rangle_{L^2\Lambda^{k-1}} & -\langle\mathbf{d}_h^{k-1}\rho_h,\mathbf{d}_h^{k-1}\tau_h\rangle_{L^2\Lambda^k} & =0\\ \langle\mathbf{d}_h^{k-1}\sigma_h^{\rm c},\mathbf{d}_h^{k-1}\varpi_h\rangle_{L^2\Lambda^k} & & =\langle\mathbf{f},\mathbf{d}_h^{k-1}\varpi_h\rangle_{L^2\Lambda^k}\end{array}\right..\tag{4.19}$$

Then

$$\zeta_h^{\rm c}=\zeta_h,\ \sigma_h^{\rm c}=\sigma_h,\ \text{and}\ \omega_h^{\rm c}=\mathbf{d}_h^{k-1}\rho_h+\delta_{k+1}\varphi_h.\tag{4.20}$$

Proof The existence of solutions to (4.18) and (4.19) is easy to verify, where ζ_h and σ_h are uniquely determined, φ_h is uniquely determined up to $\mathscr{N}(\delta_{k+1},\mathbf{W}^*_{h0}\Lambda^{k+1})$, and ρ_h is uniquely determined up to $\mathscr{N}(\mathbf{d}_h^{k-1},\mathbf{W}_h^{\rm nc}\Lambda^{k-1})$. By the Hodge decomposition of $\mathscr{P}_0\Lambda^k(\mathscr{G}_h)$, we can decompose $\omega_h\in\mathscr{P}_0\Lambda^k(\mathscr{G}_h)$ to $\omega_h^{\rm c}=\iota_h^{\rm c}+\delta_{k+1}\varphi_h^{\rm c}+\mathbf{d}_h^{k-1}\rho_h^{\rm c}$ with $\iota_h^{\rm c}\in\mathfrak{H}_h\Lambda^k$, $\varphi_h^{\rm c}\in\mathbf{W}^*_{h0}\Lambda^{k+1}$ and $\rho_h^{\rm c}\in\mathbf{W}_h^{\rm nc}\Lambda^{k-1}$, and $\mathbf{d}_h^{k-1}\rho_h^{\rm c}$ and $\delta_{k+1}\varphi_h^{\rm c}$ are uniquely determined. We can similarly write $\mu_h=\chi_h+\delta_{k+1}\psi_h+\mathbf{d}_h^{k-1}\varpi_h$. Substituting the decompositions of $\omega_h^{\rm c}$ and μ_h into (4.13) leads to subsystems (4.18) and (4.19), and further (4.20). $\square$

Noting that both (4.18) and (4.19) are each a saddle problem whose solution is not unique, we are now to further decompose them to series of semi positive definite problems to solve.

Lemma 39 *Let $(\zeta_h^{\lrcorner},\xi_h^{\lrcorner},\varphi_h^{\lrcorner})$ be a solution of the sequence of problems below:*

1. *find $\zeta_h^{\lrcorner}\in\mathbf{W}^*_{h0}\Lambda^{k+1}$, such that*
$$\langle\delta_{k+1}\zeta_h^{\lrcorner},\delta_{k+1}\psi_h\rangle_{L^2\Lambda^k}=\langle\mathbf{f},\delta_{k+1}\psi_h\rangle_{L^2\Lambda^k},\ \ \forall\,\psi_h\in\mathbf{W}^*_{h0}\Lambda^{k+1};\tag{4.21}$$
2. *find $\xi_h^{\lrcorner}\in\mathbf{W}_h^{\rm nc}\Lambda^k$, such that*
$$\langle\mathbf{d}_h^k\xi_h^{\lrcorner},\mathbf{d}_h^k v_h\rangle_{L^2\Lambda^{k+1}}=\langle\delta_{k+1}\zeta_h^{\lrcorner},v_h\rangle_{L^2\Lambda^k},\ \ \forall\,v_h\in\mathbf{W}_h^{\rm nc}\Lambda^k;\tag{4.22}$$
3. *find $\varphi_h^{\lrcorner}\in\mathbf{W}^*_{h0}\Lambda^{k+1}$, such that*
$$\langle\delta_{k+1}\varphi_h^{\lrcorner},\delta_{k+1}\eta_h\rangle_{L^2\Lambda^k}=\langle\mathbf{d}_h^k\xi_h^{\lrcorner},\eta_h\rangle_{L^2\Lambda^{k+1}},\ \ \forall\,\eta_h\in\mathbf{W}^*_{h0}\Lambda^{k+1}.\tag{4.23}$$

Let (ζ_h,φ_h) be a solution of (4.18). Then
$$\zeta_h=\frac{(-1)^{nk}}{n-k}\kappa_h^\delta(\delta_{k+1}\zeta_h^{\lrcorner})+\mathbf{d}_h^k\xi_h^{\lrcorner},\quad\textit{and}\ \ \delta_{k+1}\varphi_h=\delta_{k+1}\varphi_h^{\lrcorner}.\tag{4.24}$$

Proof Evidently, $(\zeta_h^{\lrcorner},\xi_h^{\lrcorner},\varphi_h^{\lrcorner})$ exists and is unique up to $\mathscr{N}(\delta_{k+1},\mathbf{W}^*_{h0}\Lambda^{k+1})\times\mathscr{N}(\mathbf{d}_h^k,\mathbf{W}^{\rm nc}\Lambda^k)\times\mathscr{N}(\delta_{k+1},\mathbf{W}^*_{h0}\Lambda^{k+1})$, further $\delta_{k+1}\zeta_h^{\lrcorner}=\delta_{k+1}\zeta_h$. Since $\langle\mathbf{P}_h^{k+1}\zeta_h,\mathbf{P}_h^{k+1}\eta_h\rangle_{L^2\Lambda^{k+1}}=\langle\delta_{k+1}\xi_h,\delta_{k+1}\eta_h\rangle_{L^2\Lambda^k}$ for any $\eta_h\in\mathbf{W}^*_h\Lambda^{k+1}$, it holds that $\mathbf{P}_h^{k+1}\zeta_h$ is orthogonal to $\mathscr{N}(\delta_{k+1},\mathbf{W}^*_{h0}\Lambda^{k+1})$, and thus $\mathbf{P}_h^{k+1}\zeta_h\in\mathscr{R}(\mathbf{d}_h^k,\mathbf{W}_h^{\rm nc}\Lambda^k)$. Namely, there exists a $\xi_h^{\lrcorner}\in\mathbf{W}_h^{\rm nc}\Lambda^k$, such that $\zeta_h=(\zeta_h-\mathbf{P}_h^{k+1}\zeta_h)+\mathbf{d}_h^k\xi_h^{\lrcorner}$. As for any $v_h\in\mathbf{W}_h^{\rm nc}\Lambda^k$, $\langle\delta_{k+1}\zeta_h,v_h\rangle_{L^2\Lambda^k}=\langle\zeta_h,\mathbf{d}_h^k v_h\rangle_{L^2\Lambda^{k+1}}$, it holds further that, with $\mathbf{d}_h^k v_h$ being piecewise constant, $\langle\mathbf{d}_h^k\xi_h^{\lrcorner},\mathbf{d}_h^k v_h\rangle_{L^2\Lambda^{k+1}}=\langle\delta_{k+1}\zeta_h^{\lrcorner},v_h\rangle_{L^2\Lambda^k}$. It follows by the homotopy formula that $\zeta_h=\frac{(-1)^{nk}}{n-k}\kappa_h^\delta(\delta_{k+1}\zeta_h^{\lrcorner})+\mathbf{d}_h^k\xi_h^{\lrcorner}$. Then $\langle\delta_{k+1}\varphi_h,\delta_{k+1}\eta_h\rangle_{L^2\Lambda^k}=\langle\mathbf{P}_h^{k+1}\zeta_h^{\rm c},\mathbf{P}_h^{k+1}\eta_h\rangle_{L^2\Lambda^{k+1}}=\langle\mathbf{d}_h^k\xi_h^{\lrcorner},\eta_h\rangle_{L^2\Lambda^{k+1}}$ for $\eta_h\in\mathbf{W}^*_{h0}\Lambda^{k+1}$, and it thus follows that $\delta_{k+1}\varphi_h=\delta_{k+1}\varphi_h^{\lrcorner}$. The proof is completed. $\square$

Similarly we have the decomposition of (4.19).

Lemma 40 *Let $(\sigma_h^{\lrcorner},\iota_h^{\lrcorner},\rho_h^{\lrcorner})$ be a solution of the sequence of problems below:*

1. *find $\sigma_h^{\lrcorner}\in\mathbf{W}_h^{\rm nc}\Lambda^{k-1}$, such that*
$$\langle\mathbf{d}_h^{k-1}\sigma_h^{\lrcorner},\mathbf{d}_h^{k-1}\varpi_h\rangle_{L^2\Lambda^k}=\langle\mathbf{f},\mathbf{d}_h^{k-1}\varpi_h\rangle_{L^2\Lambda^k},\ \ \forall\,\varpi_h\in\mathbf{W}_h^{\rm nc}\Lambda^{k-1};\tag{4.25}$$

2. *find* $\iota_h^{\lrcorner} \in \mathbf{W}_{h0}^{*}\Lambda^k$, *such that*

$$\langle \delta_k \iota_h^{\lrcorner}, \delta_k \chi_h \rangle_{L^2\Lambda^{k-1}} = \langle \mathbf{d}^{k-1}\sigma_h^{\lrcorner}, \chi_h \rangle_{L^2\Lambda^k}, \ \ \forall \chi_h \in \mathbf{W}_{h0}^{*}\Lambda^k; \tag{4.26}$$

3. *find* $\rho_h^{\lrcorner} \in \mathbf{W}_h^{\rm nc}\Lambda^{k-1}$, *such that*

$$\langle \mathbf{d}_h^{k-1}\rho_h^{\lrcorner}, \mathbf{d}_h^{k-1}\tau_h \rangle_{L^2\Lambda^k} = \langle \delta_k \iota_h^{\lrcorner}, \tau_h \rangle_{L^2\Lambda^{k-1}}, \ \ \forall \tau_h \in \mathbf{W}_h^{\rm nc}\Lambda^{k-1}. \tag{4.27}$$

Let (σ_h, ρ_h) *be one solution of* (4.19). *Then*

$$\sigma_h = \frac{1}{k}\kappa_h(\mathbf{d}_h^{k-1}\sigma_h^{\lrcorner}) + \delta_k \iota_h^{\lrcorner}, \quad \text{and} \ \ \mathbf{d}_h^{k-1}\sigma_h = \mathbf{d}_h^{k-1}\sigma_h^{\lrcorner}. \tag{4.28}$$

Remark 41 *It is illustrated that the system* (4.13)*, as well as* (4.5) *and* (4.3)*, can be transferred to a series of semi positive definite problems to solve. Particularly, these systems can be solved without knowledge of* $\mathfrak{H}_h\Lambda^k$*, which consists of globally supported functions and which cannot generally be figured out. A decomposition similar to Lemma 38 can be carried out onto* (4.5) *without the aid of* $\mathbf{W}_h^{\rm nc}\Lambda^k$ *and onto* (4.8) *without the aid of* $\mathbf{W}_{h0}^{*}\Lambda^k$*. However, the further decomposition of* (4.18) *and* (4.19) *will rely on the combinational utilization of* $\mathbf{W}_h^{\rm nc}\Lambda^k$ *and* $\mathbf{W}_{h0}^{*}\Lambda^k$ *together.*

5. Concluding remarks

The basis of the nonconforming finite element exterior calculus in this paper is the unified construction of finite element spaces for $H\Lambda^k$ in $\mathbb{R}^n$, extending the Crouzeix-Raviart paradigm to differential forms. These spaces are not only conceptual objectives, but also practical discretization tools. Beyond error estimation as usual, differences from existing classical schemes are preliminarily demonstrated using eigenvalue problems as examples, and can be further investigated through additional applications, for instance where a locally defined stable interpolator matters. Actually, the role of a locally-defined stable interpolator used to be illustrated by the correct computation of the convex variational problems (Ortner, 2011). This paper focuses on pure Dirichlet and pure Neumann boundary conditions. It is noteworthy that mixed boundary conditions have recently been investigated in Licht (2019a),Christiansen & Licht (2020), and Licht (2017). The new approach also works for that and can be discussed in future.

A new approach to impose inter-cell continuity is indicated, and finite element spaces can be constructed in future for various problems by this new approach. This approach also suggests potential extensions to non-simplicial meshes, as well as nonstandard and nonconforming meshes which will be discussed in future. Inspired by Lee & Winther (2018), discretization scheme for the Hodge Laplace problem with local derivatives and local coderivatives will be studied in future. The present approach is to be generalized to the primal discretization of Hodge-Laplace problems which needs nonconforming finite element spaces with proper continuity.

Recently, in two and three dimensions, discrete Helmholtz decompositions have been explored not only for piecewise constant but also for piecewise affine vector and tensor fields (Bringmann et al., 2024); it is intriguing to observe that the non-Ciarlet type finite element spaces of Fortin & Soulie (1983) and Zhang (2021) have been utilized as a basis therein. The generalization of the results presented in this manuscript to higher-degree vector and tensor fields in higher dimensions will be discussed in future.

The notion of reconstructing and preserving adjoint relation is emphasized in the present paper. Its basic role can be recognized via the duality-based argument designed to derive uniform discrete Poincaré inequalities leveraging the adjoint relationship between $\mathbf{d}$ and δ, which formulates some analogue of the closed range theorem; a quantifiable version of the closed range theorem is given in Section B for a comparison. Further, relevant to the equivalences established in Marini (1985) between the Crouzeix-Raviart element discretization and the Raviart-Thomas element discretization for Poisson equations, the equivalence between the conforming and nonconforming finite element schemes on the Hodge Laplace problem in Section 4 is the generalization of Marini (1985) with new interpretations. This novel notion can be expected to find more other applications in future.

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A. Proofs of Lemmas 23, 24 and 25

Proof of Lemma 23 Decompose $\mathbf{W}_h^{\rm nc}\Lambda^k=\mathscr{N}(\mathbf{d}_h^k,\mathbf{W}_h^{\rm nc}\Lambda^k)\oplus^\perp(\mathbf{W}_h^{\rm nc}\Lambda^k)^\lrcorner$, orthogonal in $L^2\Lambda^k(\Omega)$. Given $\sigma_h\in(\mathbf{W}_h^{\rm nc}\Lambda^k)^\lrcorner$, decompose orthogonally $\sigma_h=\mathring{\sigma}_h+\sigma_h^\lrcorner$, such that $\mathring{\sigma}_h\in\mathscr{P}_0\Lambda^k(\mathscr{G}_h)$ and $\sigma_h^\lrcorner\in\bigoplus_{T\in\mathscr{G}_h}E_T^\Omega\kappa_T(\mathscr{P}_0\Lambda^{k+1}(T))$. As $\mathscr{N}(\mathbf{d}_h^k,\mathbf{W}_h^{\rm nc}\Lambda^k)\subset\mathscr{P}_0\Lambda^k(T)$, we have further $\sigma_h^\lrcorner$ is orthogonal to $\mathscr{N}(\mathbf{d}_h^k,\mathbf{W}_h^{\rm nc}\Lambda^k)$; therefore, $\mathring{\sigma}_h$ is orthogonal to $\mathscr{N}(\mathbf{d}_h^k,\mathbf{W}_h^{\rm nc}\Lambda^k)$, and further $\mathring{\sigma}_h\in\mathscr{R}(\delta_{k+1},\mathbf{W}_{h0}^*\Lambda^{k+1})$ by Theorem 27. Decompose $\mathbf{W}_{h0}^*\Lambda^{k+1}=\mathscr{N}(\delta_{k+1},\mathbf{W}_{h0}^*\Lambda^{k+1})\oplus^\perp(\mathbf{W}_{h0}^*\Lambda^{k+1})^\lrcorner$. Then $\mathscr{R}(\delta_{k+1},\mathbf{W}_{h0}^*\Lambda^{k+1})=\mathscr{R}(\delta_{k+1},(\mathbf{W}_{h0}^*\Lambda^{k+1})^\lrcorner)$. Therefore,

$$
\begin{aligned}
\|\mathring{\sigma}_h\|_{L^2\Lambda^k}&=\sup_{\mu_h\in(\mathbf{W}_{h0}^*\Lambda^{k+1})^\lrcorner}\frac{\langle\mathring{\sigma}_h,\delta_{k+1}\mu_h\rangle_{L^2\Lambda^k}}{\|\delta_{k+1}\mu_h\|_{L^2\Lambda^k}}=\sup_{\mu_h\in(\mathbf{W}_{h0}^*\Lambda^{k+1})^\lrcorner}\frac{\langle\sigma_h^\lrcorner,\delta_{k+1}\mu_h\rangle_{L^2\Lambda^k}+\langle\mathbf{d}_h^k\sigma_h^\lrcorner,\mu_h\rangle_{L^2\Lambda^{k+1}}}{\|\delta_{k+1}\mu_h\|_{L^2\Lambda^k}}\\
&\leqslant\|\sigma_h^\lrcorner\|_{L^2\Lambda^k}+\|\mathbf{d}_h^k\sigma_h^\lrcorner\|_{L^2\Lambda^{k+1}}\sup_{\mu_h\in(\mathbf{W}_{h0}^*\Lambda^{k+1})^\lrcorner}\frac{\|\mu_h\|_{L^2\Lambda^{k+1}}}{\|\delta_{k+1}\mu_h\|_{L^2\Lambda^k}}\\
&\leqslant\|\mathbf{d}_h^k\sigma_h\|_{L^2\Lambda^{k+1}}\mathsf{pic}(\mathbf{d}_h^k,\mathscr{P}_1^-\Lambda^k(\mathscr{G}_h))+\|\mathbf{d}_h^k\sigma_h^\lrcorner\|_{L^2\Lambda^{k+1}}\mathsf{pic}(\delta_{k+1},\mathbf{W}_{h0}^*\Lambda^{k+1}).
\end{aligned}
$$

Then $\|\sigma_h\|_{L^2\Lambda^k}\leqslant\|\mathring{\sigma}_h\|_{L^2\Lambda^k}+\|\sigma_h^\lrcorner\|_{L^2\Lambda^k}\leqslant\|\mathbf{d}_h^k\sigma_h\|_{L^2\Lambda^{k+1}}(2\mathsf{pic}(\mathbf{d}_h^k,\mathscr{P}_1^-\Lambda^k(\mathscr{G}_h))+\mathsf{pic}(\delta_{k+1},\mathbf{W}_{h0}^*\Lambda^{k+1}))$. This completes the proof. □

Remark 42 *No continuous problem or Sobolev space is used as a bridge here, and this is a direct relation based on the discrete adjoint connection between $\mathbf{W}_{h0}^*\Lambda^{k+1}$ and $\mathbf{W}_h^{\rm nc}\Lambda^k$.*

Lemma 43 *There exists a constant $C_{k,n}$, depending on the regularity of T, such that*

$$\|\mu\|_{L^2\Lambda^k(T)}\leqslant C_{k,n}h_T\|\mathbf{d}^k\mu\|_{L^2\Lambda^{k+1}(T)},\ \ \textit{for}\ \ \mu\in\kappa_T(\mathscr{P}_0\Lambda^{k+1}(T)).\tag{A.1}$$

Proof Given $\mu=\sum_{\alpha\in\mathbb{IX}_{k+1,n}}C_\alpha\left(\sum_{j=1}^{k+1}(-1)^{j+1}\tilde{x}^{\alpha_j}\,\mathrm{dx}^{\alpha_1}\wedge\mathrm{dx}^{\alpha_2}\wedge\cdots\wedge\mathrm{dx}^{\alpha_{j-1}}\wedge\mathrm{dx}^{\alpha_{j+1}}\wedge\cdots\wedge\mathrm{dx}^{\alpha_{k+1}}\right)$,

$$
\begin{aligned}
|\mu|^2_{H^1\Lambda^k(T)} &= \left\|\sum_{\alpha\in\mathbb{IX}_{k,n}}C_\alpha\sum_{j=1}^{k+1}(-1)^{j+1}\nabla\tilde{x}^{\alpha_j}\,\mathrm{dx}^{\alpha_1}\wedge\cdots\wedge\mathrm{dx}^{\alpha_{j-1}}\wedge\mathrm{dx}^{\alpha_{j+1}}\wedge\cdots\wedge\mathrm{dx}^{\alpha_{k+1}}\right\|^2_{L^2\Lambda^k(T)}\\
&=\Big\langle\sum_{\alpha\in\mathbb{IX}_{k,n}}C_\alpha\sum_{j=1}^{k+1}(-1)^{j+1}\nabla\tilde{x}^{\alpha_j}\,\mathrm{dx}^{\alpha_1}\wedge\cdots\wedge\mathrm{dx}^{\alpha_{j-1}}\wedge\mathrm{dx}^{\alpha_{j+1}}\wedge\cdots\wedge\mathrm{dx}^{\alpha_{k+1}},\\
&\qquad\sum_{\alpha'\in\mathbb{IX}_{k,n}}C_{\alpha'}\sum_{i=1}^{k+1}(-1)^{i+1}\nabla\tilde{x}^{\alpha'_i}\,\mathrm{dx}^{\alpha'_1}\wedge\cdots\wedge\mathrm{dx}^{\alpha'_{i-1}}\wedge\mathrm{dx}^{\alpha'_{i+1}}\wedge\cdots\wedge\mathrm{dx}^{\alpha'_{k+1}}\Big\rangle_{L^2\Lambda^k(T)}\\
&=\sum_{\alpha\in\mathbb{IX}_{k,n}}\sum_{\alpha'\in\mathbb{IX}_{k,n}}C_\alpha C_{\alpha'}\sum_{j=1}^{k+1}\sum_{i=1}^{k+1}(-1)^{j+i}e^{\alpha_j}\cdot e^{\alpha_i}\Big\langle\mathrm{dx}^{\alpha_1}\wedge\cdots\wedge\mathrm{dx}^{\alpha_{j-1}}\wedge\mathrm{dx}^{\alpha_{j+1}}\wedge\cdots\wedge\mathrm{dx}^{\alpha_{k+1}},\\
&\qquad\mathrm{dx}^{\alpha'_1}\wedge\cdots\wedge\mathrm{dx}^{\alpha'_{i-1}}\wedge\mathrm{dx}^{\alpha'_{i+1}}\wedge\cdots\wedge\mathrm{dx}^{\alpha'_{k+1}}\Big\rangle_{L^2\Lambda^k(T)}=(k+1)|T|\sum_\alpha C_\alpha^2,
\end{aligned}
$$

and $\left\|\mathbf{d}^k\mu\right\|^2_{L^2\Lambda^{k+1}(T)}=(k+1)^2\left\|\sum_\alpha C_\alpha\,\mathrm{dx}^{\alpha_1}\wedge\mathrm{dx}^{\alpha_2}\wedge\cdots\wedge\mathrm{dx}^{\alpha_{k+1}}\right\|^2_{L^2\Lambda^{k+1}(T)}=(k+1)^2|T|\sum_\alpha C_\alpha^2.$ Namely

$$\|\mathbf{d}^k\mu\|_{L^2\Lambda^{k+1}(T)}=\sqrt{k+1}|\mu|_{H^1\Lambda^k(T)}.$$

Therefore, by noting that $\int_T\tilde{x}^j=0$, with a constant C_n depending on the regularity of T, we obtain

$$\|\mu\|_{L^2\Lambda^k(T)}\leqslant C_nh_T|\mu|_{H^1\Lambda^k(T)}=C_n(k+1)^{-1/2}h_T\|\mathbf{d}^k\mu\|_{L^2\Lambda^{k+1}(T)}.$$

This completes the proof. □

Proof of Lemma 24 Evidently,

$$
\begin{aligned}
\mathsf{pic}(\mathbf{d}^k_h,\mathscr{P}_1^-\Lambda^k(\mathscr{G}_h)) &= \sup_{\tau_h\in\bigoplus_{T\in\mathscr{G}_h}E_T^\Omega\kappa_T(\mathscr{P}_0\Lambda^{k+1}(T))}\frac{\|\tau_h\|_{L^2\Lambda^k}}{\|\mathbf{d}^k_h\tau_h\|_{L^2\Lambda^{k+1}}}\\
&=\max_{T\in\mathscr{G}_h}\sup_{\tau\in\kappa_T(\mathscr{P}_0\Lambda^{k+1}(T))}\frac{\|\tau\|_{L^2\Lambda(T)}}{\|\mathbf{d}^k\tau\|_{L^2\Lambda^{k+1}(T)}}.
\end{aligned}
\tag{A.2}
$$

By Lemma 43 and (A.2), $\mathsf{pic}(\mathbf{d}^k_h,\mathscr{P}_1^-\Lambda^k(\mathscr{G}_h))$ is of $\mathscr{O}(h)$ order. □

Proof of Lemma 25 By virtue of Lemma 23 and Remark 17, $\mathsf{pic}(\delta_{k+1},\mathbf{W}^*_{h0}\Lambda^{k+1})$ is controlled by $\mathsf{pic}(\mathbf{d}^k_h,\mathbf{W}^{\mathrm{nc}}_h\Lambda^k)$ the same way. Further by Lemma 24, we obtain Lemma 25. □

B. A quantifiable closed range theorem

In this part, we establish a quantifiable version of the classical closed range theorem, in order to show how Lemma 25 can be viewed as a discrete analogue of the closed range theorem.

Let $\mathbf{X}$ and $\mathbb{Y}$ be two Hilbert spaces with respective inner products $\langle\cdot,\cdot\rangle_{\mathbf{X}}$ and $\langle\cdot,\cdot\rangle_{\mathbb{Y}}$, and let $(\mathbf{T},\mathbf{D}):\mathbf{X}\to\mathbb{Y}$ be an unbounded linear operator, $\mathbf{D}$ being the domain dense in $\mathbf{X}$. The adjoint operator of $(\mathbf{T},\mathbf{D})$, denoted by $(\mathbf{T}^*,\mathbf{D}^*)$, is defined by

$$\langle\mathbf{T}^*\mathbb{w},\mathbf{v}\rangle_{\mathbf{X}}=\langle\mathbb{w},\mathbf{T}\mathbf{v}\rangle_{\mathbb{Y}},\quad\forall\,\mathbf{v}\in\mathbf{D},\tag{B.1}$$

and the domain $\mathbf{D}^*$ consists of such $\mathbb{w}\in\mathbb{Y}$ that there exists an element in $\mathbf{X}$ taken as $\mathbf{T}^*\mathbb{w}$ to satisfy (B.1). The closed range theorem (cf. Arnold (2018); Yosida (2012); Kato (2013); Brezis (2010) and other textbooks) asserts that

$$\mathscr{R}(\mathbf{T},\mathbf{D})\text{ is closed}\iff\mathscr{R}(\mathbf{T}^*,\mathbf{D}^*)\text{ is closed.}\tag{B.2}$$

It further follows by Lemma 21 that

$$\mathsf{pic}(\mathbf{T},\mathbf{D})<\infty\iff\mathsf{pic}(\mathbf{T}^*,\mathbf{D}^*)<\infty.\tag{B.3}$$

The theorem below further gives a preciser quantification of the closed range theorem.

Theorem 44 *For $(\mathbf{T},\mathbf{D}):\mathbf{X}\to\mathbb{Y}$ and $(\mathbb{T},\mathbb{D}):\mathbb{Y}\to\mathbf{X}$ a pair of closed densely defined adjoint operators,*

$$\mathsf{pic}(\mathbf{T},\mathbf{D})=\mathsf{pic}(\mathbb{T},\mathbb{D}).\tag{B.4}$$

Proof Recalling the Helmholtz decomposition $\mathbf{X}=\mathscr{N}(\mathbf{T},\mathbf{D})\oplus^{\perp}\overline{\mathscr{R}(\mathbb{T},\mathbb{D})}$, we have

$$\mathbf{D}^{\lrcorner}=\mathbf{D}\cap(\mathscr{N}(\mathbf{T},\mathbf{D}))^{\perp}=\mathbf{D}\cap\overline{\mathscr{R}(\mathbb{T},\mathbb{D})}.\tag{B.5}$$

Therefore, provided that $0<\mathsf{pic}(\mathbb{T},\mathbb{D})<\infty$ and thus $\overline{\mathscr{R}(\mathbb{T},\mathbb{D})}=\mathscr{R}(\mathbb{T},\mathbb{D})$, given $\mathbf{v}\in\mathbf{D}^{\lrcorner}$, there exists a $\mathbb{w}\in\mathbb{D}^{\lrcorner}$, such that $\mathbf{v}=\mathbb{T}\mathbb{w}$, then $\|\mathbb{w}\|_{\mathbb{Y}}\leqslant\mathsf{pic}(\mathbb{T},\mathbb{D})\|\mathbf{v}\|_{\mathbf{X}}$ and

$$\|\mathbf{v}\|_{\mathbf{X}}^2=\langle\mathbf{v},\mathbf{v}\rangle_{\mathbf{X}}=\langle\mathbf{v},\mathbb{T}\mathbb{w}\rangle_{\mathbf{X}}=\langle\mathbf{T}\mathbf{v},\mathbb{w}\rangle_{\mathbb{Y}}\leqslant\|\mathbf{T}\mathbf{v}\|_{\mathbb{Y}}\|\mathbb{w}\|_{\mathbb{Y}}\leqslant\mathsf{pic}(\mathbb{T},\mathbb{D})\|\mathbf{T}\mathbf{v}\|_{\mathbb{Y}}\|\mathbf{v}\|_{\mathbf{X}}.$$

Therefore, $\|\mathbf{v}\|_{\mathbf{X}}\leqslant\mathsf{pic}(\mathbb{T},\mathbb{D})\|\mathbf{T}\mathbf{v}\|_{\mathbf{X}}$ for any $\mathbf{v}\in\mathbf{D}^{\lrcorner}$ and $\mathsf{pic}(\mathbf{T},\mathbf{D})\leqslant\mathsf{pic}(\mathbb{T},\mathbb{D})<\infty$. Similarly, $\infty>\mathsf{pic}(\mathbf{T},\mathbf{D})\geqslant\mathsf{pic}(\mathbb{T},\mathbb{D})$; note that $(\mathbf{T},\mathbf{D})$ is the adjoint operator of $(\mathbb{T},\mathbb{D})$. Namely, if one of $\mathsf{pic}(\mathbf{T},\mathbf{D})$ and $\mathsf{pic}(\mathbb{T},\mathbb{D})$ is finitely positive, then $\mathsf{pic}(\mathbf{T},\mathbf{D})=\mathsf{pic}(\mathbb{T},\mathbb{D})$.

If $\mathsf{pic}(\mathbb{T},\mathbb{D})=0$, then $\mathscr{R}(\mathbb{T},\mathbb{D})=\{0\}$ and $\mathbf{D}^{\lrcorner}=\{0\}$. It follows then $\mathsf{pic}(\mathbf{T},\mathbf{D})=0$. Finally, if one of $\mathsf{pic}(\mathbf{T},\mathbf{D})$ and $\mathsf{pic}(\mathbb{T},\mathbb{D})$ is $+\infty$, then so is the other. The proof is completed. □

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